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Asymptotic behavior of semigroups of nonlinear contractions in Hilbert spaces

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Analisi funzionale. — *Asymptotic behavior of semigroups of nonlinear contractions in Hilbert spaces.* Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Sia $S(t, x)$ un semigruppo di contrazioni non lineari definite in un sottoinsieme D di uno spazio di Hillert. Si dimostra che $S(t, x)/t$ per $t \to \infty$ tende ad un limite per ogni $x \in D$. Questo limite è indipendente da x ed è in relazione con i generatori di S.

1. INTRODUCTION

Let D be a non-empty subset of a real Hilbert space H. A mapping $T: D \to H$ is said to be a (nonlinear) contraction if $|Tx - Ty| \leq |x - y|$ for all x and y in D. A semigroup (of nonlinear contractions) on D is a function $S: [0, \infty) \times D \to D$ satisfying the following conditions:

(1.1) $S(t_1 + t_2, x) = S(t_1, S(t_2, x))$ for $t_1, t_2 \geq 0$ and $x \in D$;

(1.2) $|S(t, x) - S(t, y)| \leq |x - y|$ for $t \geq 0$ and $x, y \in D$;

(1.3) $S(0, x) = x$ for $x \in D$;

(1.4) $\lim_{t \to t_0} S(t, x) = S(t_0, x)$ for $t, t_0 \geq 0$ and $x \in D$.

The purpose of this note is to study a certain aspect of the behavior of $S(t, x)$ when $t \to \infty$.

We shall denote the closure of D by cl(D). Its convex hull will be denoted by co(D) and its convex closure by clco(D). We also define

$$\|D\| = \inf\{|x| : x \in D\} \quad \text{and} \quad D^0 = \{x \in D : |x| = \|D\|\}.$$

The identity operator (on D) will be denoted by I.

If A is a subset of $H \times H$ and $x \in H$, we define $Ax = \{y \in H : [x, y] \in A\}$ and let $D(A) = \{x \in H : Ax \neq \emptyset\}$. The range of A is defined by $R(A) = \cup \{Ax : x \in D(A)\}$. Such a set A is said to be monotone if $(y_1 - y_2, x_1 - x_2) \geq 0$ for all $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$.

A monotone set is maximal monotone if it does not admit a proper monotone extension. If A is monotone one can define, for each $r > 0$, a single-valued function $J_r: R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. We also define the Yosida approximation of A, $A_r: R(I + rA) \to H$, by $A_r = (I - J_r)/r$.

(*) Nella seduta del 29 giugno 1974.

Suppose that

$$\text{(1.5)} \qquad R(I + rA) \supset D(A) \qquad \text{for all } r > 0.$$

Then there exists a semigroup S on cl (D (A)) such that for each $x \in D(A)$ and $t \geq 0$

$$\text{(1.6)} \qquad S(t, x) = \lim_{n\to\infty} J_{t/n}^n x = \lim_{n\to\infty} \left(I + \frac{t}{n} A\right)^{-n} x$$

[5, p. 271]. We shall say that S is generated by $-A$ via the exponential formula (1.6).

Let A be monotone and consider the initial value problem

$$\text{(1.7)} \qquad \begin{cases} \dfrac{du}{dt} + Au \ni 0 & \text{a.e. on } (0, \infty) \\ u(0) = x \end{cases}$$

where $x \in D(A)$. Suppose that $v : [0, \infty) \to H$ is (Bochner) integrable on every interval of the form $[0, T]$, $T < \infty$. Let $u(t) = u(0) + \int_0^t v(s)\, ds$. Then $u : [0, \infty) \to H$ is absolutely continuous (on $[0, T]$), differentiable a.e. on $[0, \infty)$, and $\frac{du}{dt} = v$ a.e. on $[0, \infty)$. Such a function u is called a solution of the IVP (1.7) if $u(t) \in D(A)$ a.e. on $(0, \infty)$ and $u(t)$ satisfies (1.7). Since an absolutely continuous function which is differentiable a.e. is an indefinite integral of its derivative, we could have assumed that u is absolutely continuous and differentiable a.e. Since H is reflexive, it is sufficient, in fact, to assume that u is absolutely continuous on every interval of the form $[0, T]$.

The IVP has at most one solution. This solution is Lipschitzian on $[0, \infty)$. If u, v are two solutions of the IVP, then $|u(t) - v(t)| \leq |u(0) - v(0)|$ for all $t \in [0, \infty)$. It follows that if the IVP has a solution for each $x \in D(A)$, then a semigroup can be defined on cl (D (A)). This is said to be the semigroup generated by $-A$ via the initial value problem (1.7). If A satifies (1.5), then it must coincide with the semigroup generated by the EF (1.6) [3, p. 371]. On the other hand, if A is closed ($x_n \in D(A)$, $y_n \in Ax_n$, $x_n \to x$ and $y_n \to y$ imply that $[x, y] \in A$), then the EF semigroup is also the IVP semigroup [8, p. 252].

2. ASYMPTOTIC BEHAVIOR

At the beginning we shall study the behavior of a given semigroup without metioning possible generators. Let N denote the set of all non-negative integers, and let $\{c_n : n \in N\}$ be a sequence of real numbers which satisfy

$$\text{(2.1)} \qquad 0 < c_n \leq 1 \qquad \text{for all } n \in N;$$

$$\text{(2.2)} \qquad \sum_{i=0}^{\infty} c_i \qquad \text{diverges.}$$

In the sequel we shall denote $\sum_{i=0}^{n} c_i$ by a_n.

LEMMA 2.1. *Let* D *be a subset of a Hilbert space* H *and let* $T : D \to H$ *be a contraction. Suppose that there exists a sequence* $\{x_n : n \in N\} \subset D$ *such that*

$$x_{n+1} = (1 - c_n) x_n + c_n T x_n \quad , \quad n \in N. \tag{2.3}$$

Then $\{x_{n+1}/a_n : n \in N\}$ *converges.*

Proof. Extend T to a contraction $Q : H \to H$ [13, p. 92]. Since (2.3) holds when T is replaced by Q, $\lim_{n\to\infty} x_{n+1}/a_n = -v$ where v is the element of minumum norm in cl (R (I — Q)) [10, p. 695].

COROLLARY 2.2. *Let* D *be a subset of a Hilbert space* H. *If* $T : D \to D$ *is a contraction and* $x_0 \in D$, *then* $\lim_{n\to\infty} T^n x_0/n$ *exists* (*and is independent of* x_0).

Proof. Lemma 2.1 can be used with $c_n = 1$ for all n. (See also [9, p. 238]).

THEOREM 2.3. *Let* S *be a semigroup on a subset* D *of a Hilbert space* H. *Then* $\lim_{t\to\infty} S(t, x)/t$ *exists for each* $x \in D$ (*and is independent of* x).

Proof. Denote the mapping which assigns to each $y \in D$ the point $S(1, y) \in D$ by T. Fix a point $x \in D$. By Corollary 2.2, $\lim_{n\to\infty} S(n, x)/n = \lim_{n\to\infty} T^n x/n = -v$ exists. Let ε be positive. There is an N such that $|T^n x/n + v| < \varepsilon$ for all $n > N$. Let M satisfy $|T^n x/n| \leq M$ $(n > N)$, $|S(t, x)| \leq M$ $(0 \leq t \leq 1)$, and let t_0 be greater than $\max\{M/\varepsilon, N + 1\}$. If $t > t_0$, $[t] = n$, and $t - [t] = p$, then

$$\begin{aligned} |S(t, x)/t + v| &\leq |S(t, x)/t - S(n, x)/n| + \\ + |S(n, x)/n + v| &\leq |S(n, S(p, x))/(n+p) - S(n, x)/n| + \\ + \varepsilon &\leq |S(p, x) - x|/t + p\,|S(n, x)|/(nt) + \varepsilon \leq 2M/t + M/t + \varepsilon < 4\varepsilon, \end{aligned}$$

as required.

In order to identify the limit obtained in Theorem 2.3, we shall need a few auxiliary propositions.

LEMMA 2.4. *Let* u *be the solution* (*if it exists*) *of the* IVP (1.7). *Then*

$$|u(t) - u(0)| \leq t\,\|Ax\|. \tag{2.4}$$

Proof. We have $u(t) - u(0) = \int_0^t \frac{du}{dt}(s)\,ds$. Therefore

$$|u(t) - u(0)| \leq \int_0^t \left|\frac{du}{dt}(s)\right| ds \leq t\,\|Ax\| \quad \text{by [3, p. 369].}$$

A closed subset $B \subset H$ is said to have the minimum property [9, p. 237] if it contains a point the norm of which equals $\|\text{clco}(B)\|$.

PROPOSITION 2.5. *Let* A *be a monotone set in a Hilbert space* H. *Suppose that* $-$A *generates a semigroup* S *on* cl (D (A)) *through the initial value problem* (1.7). *If* cl (R (A)) *has the minimum property, then for each* $x \in$ cl (D (A)) $\lim_{t\to\infty} S(t, x)/t = -v$ *where* v *is the element of least norm in* cl (R (A)).

Proof. Let $x \in D(A)$, $|v| = d$ and $\varepsilon > 0$. There is $[y, z] \in A$ such that $|z - v| < \varepsilon$. We have $|S(t, x) - x| \leq |S(t, x) - S(t, y)| + |S(t, y) - y| + |y - x| \leq 2|x - y| + t\|Ay\| \leq 2|x - y| + t|z| \leq 2|x - y| + t(d + \varepsilon)$. If follows that $\limsup_{t\to\infty} |S(t, x) - x|/t \leq d$. On the other hand, $(x - S(t, x))/t = \frac{1}{t}\int_0^t -\frac{du}{dt}(s)\,ds \in \mathrm{clco}(R(A))$. Therefore $|S(t,x) - x|/t \geq d$. Thus $\lim_{t\to\infty} |x - S(t,x)|/t = d$ and $(x - S(t, x))/t \xrightarrow[t\to\infty]{} v$. The result follows.

The method of proof of this Proposition can be used to establish a result of Crandall mentioned (without proof) in [1, p. 166].

LEMMA 2.6. *Let* S *be generated by* $-$A *through the exponential formula* (1.6). *Then*

$$|S(t, x) - x| \leq t\|Ax\| \tag{2.5}$$

for each $x \in D(A)$.

Proof. We have

$$|J^n_{t/n} x - x| \leq n\,|J_{t/n} x - x| \leq n(t/n)\,\|Ax\| = t\,\|Ax\|.$$

LEMMA 2.7. *Let* S *be generated by* $-$A *through the* EF (1.6). *Then* $(x - S(t, x)/t$ *belongs to* clco (R (A)).

Proof. We have

$$(x - J^n_{t/n} x)/t = \frac{1}{n}\sum_{i=0}^{n-1} (J^i_{t/n} x - J^{i+1}_{t/n} x)/(t/n)$$

and $A_r y \in AJ_r y$. The result follows.

PROPOSITION 2.8. *Let* A *be a monotone set in a Hilbert space* H. *Suppose that* $-$A *generates a semigroup* S *on* cl (D (A)) *through the exponential formula* (1.6). *If* cl (R (A)) *has the minimum property, then for each* $x \in$ cl (D (A)) $\lim_{t\to\infty} S(t, x)/t = -v$ *where* v *is the element of least norm in* cl (R (A)).

Proof. We can mimic the proof of Proposition 2.5 with the aid of Lemmas 2.6 and 2.7.

PROPOSITION 2.9 [7, p. 385]. *Let* S *be a semigroup on a subset* D *of a Hilbert space* H. *Then there exists a semigroup on* clco (D) *which extends* S.

PROPOSITION 2.10 [2, p. 257; 6, p. 417; 7, p. 396]. *Let* S *be a semigroup defined on a closed and convex subset* C *of a Hilbert space* H. *Then there exists a unique maximal monotone* $A \subset H \times H$ *with* $\mathrm{cl}(D(A)) = C$ *such that* $-A$ *generates* S *through both the initial value problem* (1.7) *and the exponential formula* (1.6).

THEOREM 2.11. *Let* S *be a semigroup on a subset* D *of a Hilbert space* H. *Let* A *be the unique maximal monotone subset of* $H \times H$ *with* $\mathrm{cl}(D(A)) = \mathrm{clco}(D)$ *such that* $-A$ *generates the extension of* S. *Then* $\lim_{t\to\infty} S(t, x)/t = -v$ *for each* $x \in D$ *where* v *is the element of least norm in* $\mathrm{cl}(R(A))$.

Proof. We can combine Proposition 2.10 with either Propostion 2.5 or Proposition 2.8 because $\mathrm{cl}(R(A))$ is convex [12, p. 89].

COROLLARY 2.12. *Let* S *be a semigroup on a closed and convex subset* C *of a Hilbert space* H. *Let* A *be the unique maximal monotone set with* $\mathrm{cl}(D(A)) = C$ *such that* $-A$ *generates* S. *Then the element of least norm in* $\mathrm{cl}(R(A))$ *is the element of least norm in* $\mathrm{cl}(R(I - S(1, \cdot)))$.

Proof. Combine [9, Corollary 3] with Theorem 2.11.

3. THE MINIMUM PROPERTY

We turn now to the converse problem namely, given a monotone A such that $-A$ generates a semigroup S, how can $\lim_{t\to\infty} S(t, x)/t$ be related to A? By Propositions 2.5 and 2.8 it suffices to show that $\mathrm{cl}(R(A))$ has the minimum property. It is obvious, for example, that if $0 \in \mathrm{cl}(R(A))$, then $\mathrm{cl}(R(A))$ has the MP. Simple examples show, however, that $\mathrm{cl}(R(A))$ does not always possess the MP. The following result is a generalization of [9, Theorem 3].

PROPOSITION 3.1. *Let* A *be monotone. Suppose that there exists a sequence* $\{x_n : n \in N\} \subset D(A)$ *such that*

$$x_{n+1} = x_n - c_n y_n \quad , \quad n \in N, \tag{3.1}$$

where $y_n \in Ax_n$ *and* $\{c_n : n \in N\}$ *satisfies* (2.1) *and* (2.2). *If* $\{y_n : n \in N\}$ *converges, then* $\mathrm{cl}(R(A))$ *has the minimum property.*

Proof. Let $v = \lim_{n\to\infty} y_n$ and $a_n = \sum_{i=0}^{n} c_i$. Since $x_0 - x_{n+1} = \sum_{i=0}^{n} c_i y_i$, $(x_0 - x_{n+1})/a_n \to v$ and $x_{n+1}/a_n \to -v$. If $u \in D(A)$ and $w \in Au$, then $(w - y_{n+1}, u - x_{n+1}) \geq 0$. After divding by a_n and letting n tend to infinity we obtain $(w - v, v) \geq 0$. Thus $(z - v, v) \geq 0$ for all $z \in \mathrm{clco}(R(A))$ and $|v| = \|\mathrm{clco}(R(A))\|$.

PROPOSITION 3.2. *Let a monotone and closed* A *satisfy* (1.5). *If there exists a maximal monotone extension* B *of* A *with* $D(B) \subset \mathrm{cl}(D(A))$, *then* $\mathrm{cl}(R(A))$ *has the minimum property.*

Proof. Let $z \in D(B)$. For each $r > 0$ there are $x_r \in D(A)$ and $y_r \in Ax_r$ such that $z = x_r + ry_r$. Since $x_r = J_r^B z$ and $y_r = B_r z$, $\lim_{r\to 0} x_r = z$ and $\lim_{r\to\infty} y_r = B^0 z$ where $B^0 z = (Bz)^0$. Thus $z \in D(A)$ and $B^0 z \in Az$. Thus $D(B) = D(A)$ and $B^0 = A^0$. Now let v be the element of least norm in $cl(R(B))$, and let $v_n \to v$, $v_n \in Bx_n$. Then $|v| \leq |A^0 x_n| = |B^0 x_n| \leq |v_n|$. It follows that $A^0 x_n \to v$.

Although the next result is known [2, p. 244], we present a different, direct proof which does not use Brézis' variational inequality. (We do employ variants of some known ideas).

PROPOSITION 3.3. *Let a monotone and closed* A *satisfy* (1.5). *If* cl (D (A)) *is convex, then there exists a maximal monotone extension* E *of* A *with* $D(E) = D(A)$.

Proof. Denote cl (D (A)) by C and let $E = A + B$ where $B = \partial I_C$ (the subdifferential of the indicator function of C) and $D(E) = D(A)$. B is maximal monotone. If $r > 0$, $0 < t < 2/r$ and $y \in H$, then the equation $y \in x + tBx + tA_r x$, which is equivalent to $x = J_t^B(y - tA_r x)$, has a solution in C by Banach's fixed point theorem. If follows that $B + A_r$ is maximal monotone. Let $y \in H$ and let $x_r \in D(B)$ satisfy $y \in x_r + Bx_r + A_r x_r$ for each positive r. Fix $u \in D(A)$ and $w \in Bu$. Since $y - x_r - A_r x_r \in Bx_r$, we have $(w + A_r u - y + x_r + A_r x_r - A_r x_r, u - x_r) \geq 0$ and $(w + A_r u - y + u - (u - x_r), u - x_r) \geq 0$. Thus $|u - x_r|^2 \leq |u - x_r| \, |w + A_r u - y + u|$ and consequently $\{x_r\}$ is bounded. We also have $(y - x_r - A_r x_r, x_r - z) \geq 0$ for every $z \in C$. Taking $z = J_r^A x_r$ we obtain $(y - x_r - A_r x_r, rA_r x_r) \geq 0$ and $|A_r x_r|^2 \leq |A_r x_r| \, |y - x_r|$. It follows that $\{A_r x_r\}$ is bounded too. Denote $y - x_r - A_r x_r \in Bx_r$ by b_r. We have $x_r - x_s + b_r - b_s + A_r x_r - A_s x_s = 0$. Since B is monotone, it follows that $|x_r - x_s|^2 \leq -(A_r x_r - A_s x_s, x_r - x_s)$. We also have $x_r - x_s = rA_r x_r - sA_s x_s + J_r^A x_r - J_s^A x_s$. Since $A_r x_r \in AJ_r^A x_r$ and $A_s x_s \in AJ_s^A x_s$, we obtain $|x_r - x_s|^2 \leq -(A_r x_r - A_s x_s, rA_r x_r - sA_s x_s)$. Thus $\{x_r\}$ converges to $x \in C$. $|J_r^A x_r - x_r| = r|A_r x_r| \to 0$, so that $J_r^A x_r \to x$. If the convergence of $\{A_r x_r\}$ is established, then we have $x \in D(A)$ and $a = \lim_{r\to 0} A_r x_r \in Ax$. Consequently, $y = x + a + b$ where $a \in Ax$ and $b = \lim_{r\to 0} b_r \in Bx$. In fact, the convergence of $\{A_r x_r\}$ can be established by invoking [6, Lemma 2.4] or by the following argument: Let G be a maximal monotone extension of A in cl (D (A)). Then G is demi-closed and Gu is closed and convex for each $u \in D(G)$.

Let a subsequence of $\{A_r x_r = G_r x_r\}$ converge weakly to $w \in Gx$. Then $y - x - w \in Bx$. Thus $Gx \cap (y - x - Bx)$ is not empty. Let z be the unique element of least norm in this closed and convex subset of H. Since $y = b_r + x_r + G_r x_r = b + x + z$ for some $b \in Bx$, we have $(b_r - b + x_r - x + G_r x_r - z, x_r - x) = 0$ and $(G_r x_r - z, x_r - x) \leq 0$. Writing $x_r = rG_r x_r + J_r^G x_r$, we obtain $(G_r x_r - z, G_r x_r) \leq 0$. Let w be any subsequential weak limit of $\{G_r x_r\}$. Then $|w| \leq \lim\inf |G_s x_s| \leq \lim\sup |G_s x_s| \leq |z|$. It follows that $w = z$, $\{G_r x_r\}$ converges weakly to z, $|G_r x_r| \rightarrow |z|$, and finally that $\{A_r x_r\}$ converges strongly to z.

The last two Propositions can be combined to yield [4, Theorem 1] because the condition imposed there on A implies the convexity of cl (D (A)) [3, p. 382].

THEOREM 3.4. *Let a monotone and closed* A *satisfy* (1.5). *If* cl (D (A)) *is convex and* S *is the semigroup generated by* — A *on* cl (D (A)), *then for each* $x \in$ cl (D (A)) $\lim_{t\to\infty} S(t, x)/t = -v$, *where* v *is the element of least norm in* cl (R (A)).

Proof. Combine Propositions 3.2, 3.3 and 2.8 (or 2.5).

The asymptotic behavior of nonlinear contraction semigroups in Banach spaces is considered in [11].

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