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Boundedness and stability for a nonlinear third order differential equation

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Equazioni differenziali non lineari. — *Boundedness and stability for a nonlinear third order differential equation.* Nota di K. E. SWICK, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Data l'equazione $\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$ l'Autore, trova per le funzioni g, h, p condizioni sufficienti per la uniforme limitatezza e convergenza a zero delle soluzioni.

I risultati dipendono essenzialmente su disuguaglianze relative a $(1/x)\int_0^x g(u)\,du$, $h(x)x$.

1. INTRODUCTION

In this paper we investigate the behavior of solutions of the differential equation

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1.1}$$

under the basic assumptions:

(1.2) The functions g and h are continuous and real valued on the reals, a is a positive constant and p is continuous and real valued for $t \geq 0$ and all real numbers x, y and z;

(1.3) There are constants $b > 0$ and $B \geq 0$ such that $\frac{G(x)}{x} \geq b$ for $|x| > B$ where $G(x) = \int_0^x g(u)\,du$;

(1.4) There are nonnegative functions $e_1(t)$ and $e_2(t)$ such that $|p(t, x, y, z)| \leq e_1(t) + e_2(t)(x^2 + y^2 + z^2)^{1/2}$ for $t \geq 0$ and all x, y and z.

We will call the solutions of Eq. (1.1) uniform ultimately bounded, if there is $K > 0$ such that for every solution $x(t) = x(t; x_0, t_0)$ $(t_0 \geq 0)$ of Eq. (1.1) there is $T > 0$ such that $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \leq K$ for $t \geq t_0 + T$.

If (1.2)–(1.4) are satisfied where $e_2(t) = 0$ $t \geq 0$ and $e_1(t) \leq B_0$ for some $B_0 \geq 0$, then it was shown in [7] that if $h(x)\,\mathrm{sgn}\,x \geq \eta$ and $h'(x) \leq c$ for $|x| > B$ where $0 < c < ab$ and $\eta > c/2a$, then the solutions of Eq. (1.1) are uniform ultimately bounded. Previously, boundedness and asymptotic behavior of solutions of this equation had been investigated by Ezeilo [1]–[3].

(*) Nella seduta del 20 aprile 1974.

Haas [4], Lalli [5], Müller [6], Swick [8], and Voráček [10]. Recently Tejumola [9] investigated boundedness of solutions of the equation

$$\dddot{x} + f(x, \dot{x}, \ddot{x})\,\ddot{x} + g(x, \dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}),$$

although he required much more severe restrictions on $g(x, \dot{x})$ than was required in the preceding investigations of Eq. (1.1).

All of these results have contained the restriction that $h(x)$ be bounded by a linear function. In fact, in each instance this restriction has taken the form $\frac{h(x)}{x} > 0$ $|x| > B$ and either $\frac{h(x)}{x} \leq c$ $|x| > B$ or $h'(x) \leq c$ where $c < ab$. It will be shown here that for a certain large collection of equations, uniform ultimate boundedness and convergence to zero of the solutions of Eq. (1.1) can be obtained when $h(x)$ is much larger than the bounds indicated in the previous results.

To accomplish this goal, we will look for a Liapunov function of the form $V = \int_0^x aG(u) - h(u)\,du + q_0(x, y, z)$ where q_0 is a quadratic form in x, y and z. The only restriction on h for positive definiteness of this function will be that $\frac{h(x)}{x} \leq \frac{aG(x)}{x}$. With the proper choice of q_0, negative definiteness of $\dot{V}$ will depend on an inequality involving h and G.

Define the function $Q(x, \alpha)$ by

$$Q(x, \alpha) = a\left[a\,\frac{G(x)}{x} - 2\,\frac{h(x)}{x}\right] - \left[\frac{a^2}{2} + \frac{G(x)}{x} - \alpha\,\frac{h(x)}{x}\right]^2.$$

THEOREM 1. *Let* (1.2)–(1.4) *be satisfied for* $B \geq 0$. *If there are positive constants* B_0 *and* $\alpha > \frac{2(a^2+b)}{a(a^2+2b)}$ *such that*

$$\inf_{|x|>B} Q(x, \alpha) > 0. \tag{1.5}$$

$$e_1(t) \leq B_0 \quad t \geq 0. \tag{1.6}$$

then there is $\varepsilon > 0$ *such that if* $0 \leq e_2(t) \leq \varepsilon$ $t \geq 0$, *then every solution* $x(t : x_0, t_0)$ $(t_0 \geq 0)$ *exists for* $t \geq t_0$ *and the solutions of Eq.* (1.1) *are uniform ultimately bounded.*

We note that in order that (1.5) be satisfied, $h(x)$ must satisfy $\frac{h(x)}{x} \geq c$ $|x| > B$ for some $c > 0$.

THEOREM 2. *Let* (1.2)–(1.4) *be satisfied where* $B = 0$. *If there is an* $\alpha > \frac{2(a^2+b)}{a(a^2+2b)}$ *such that*

$$\inf_{x \neq 0} Q(x, \alpha) > 0. \tag{1.7}$$

$$\int_0^\infty e_1(t) + e_2(t)\,dt < \infty. \tag{1.8}$$

then there is $\varepsilon > 0$ *such that if* $0 \leq e_2(t) \leq \varepsilon$ $t \geq 0$, *then every solution* $x(t) = x(t; x_0, t_0)$ $t_0 \geq 0$ *satisfies*

$$x(t) \to 0 \quad , \quad \dot{x}(t) \to 0 \quad , \quad \ddot{x}(t) \to 0 \qquad as \quad t \to \infty .$$

Since the objective here is to allow $h(x)$ to be as large as possible, one would not expect these inequalities to reduce to the Routh-Hurwitz conditions $ab > c > 0$ for the linear equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$. It can be easily shown however, that in this case if $ab > 2c$, then the hypotheses of Theorems 1 and 2 are satisfied.

It is clear that in some sense $h(x)$ must remain "close" to $\frac{1}{\alpha} G(x)$ in order that the inequality in (1.5) or (1.7) be satisfied. In order to investigate some of the implications of this inequality, we set $f(x) = \frac{a}{d} G(x) - h(x)$ and look for an inequality for $f(x)$ which will satisfy (1.5) or (1.7). The following result gives one such answer.

THEOREM 3. *If there exist constants* $\varepsilon > 0$, $B \geq 0$ *and* $d > 2$ *such that*

$$(1.9) \qquad \frac{G(x)}{x} > \frac{a^2(d-1) + \varepsilon d(d-2)}{d(d-2)} \qquad |x| > B,$$

$$(1.10) \qquad \frac{-a^2}{d^2} \left\{ \left[d(d-2) \frac{G(x)}{x} - a^2(d-1) - \varepsilon \right]^{1/2} + \frac{a}{2}(d-2) \right\} \leq \frac{f(x)}{x} \leq$$

$$\leq \frac{a^2}{d^2} \left\{ \left[d(d-2) \frac{G(x)}{x} - a^2(d-1) - \varepsilon \right]^{1/2} - \frac{a}{2}(d-2) \right\} \qquad |x| > B.$$

then $\inf\limits_{|x|>B} Q\left(x, \frac{d}{a}\right) > 0$ *and* $\alpha = \frac{d}{a} > \frac{2(a^2+b)}{a(a^2+2b)}$.

We now consider the equation

$$(1.11) \qquad \dddot{x} + a\ddot{x} + g(x)\dot{x} + \frac{a}{d} G(x) - f(x) = p(t, x, \dot{x}, \ddot{x})$$

which can be obtained from Eq. (1.1) by setting $f(x) = \frac{a}{d} G(x) - h(x)$. The following is a direct result of Theorems 1 and 3.

THEOREM 4. *Let* (1.2)–(1.4) *be satisfied for* $B \geq 0$. *If* $d > 2$ *and there is* $B_0 > 0$ *such that*

$$(1.12) \qquad e_1(t) \leq B_0 \qquad t \geq 0.$$

$$(1.13) \qquad \liminf_{|x| \to \infty} \frac{a^4(d-2)}{2d^3} \frac{G(x)}{x} - \frac{f^2(x)}{x^2} = +\infty.$$

then there is $\varepsilon > 0$ *such that if* $0 \leq e_2(t) \leq \varepsilon$, $t \geq 0$, *then every solution* $x(t; x_0, t_0)$ *exists for* $t \geq t_0$, *and the solutions of Eq.* (1.11) *are uniform ultimately bounded.*

2. Proof of Theorem 1.

Eq. (1.1) is equivalent to the system of equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z - ay - \mathrm{G}(x) \\ \dot{z} &= -h(x) + p(t, x, \dot{x}, \ddot{x}). \end{aligned} \tag{2.1}$$

Define the function $\mathrm{V} = \mathrm{V}(x, y, z)$ as

$$\mathrm{V} = 2\int_0^x [a\mathrm{G}(u) - h(u)]\,\mathrm{d}u + \frac{a^3}{2}x^2 + ay^2 + \alpha z^2 + a^2 xy - 2\,axz - 2\,yz$$

where $\alpha > 0$.

Since $\inf_{|x|>\mathrm{B}} \mathrm{Q}(x, \alpha) > 0$, it follows from an examination of $\mathrm{Q}(x, \alpha)$ that $\frac{a\mathrm{G}(x)}{x} \geq \frac{2\,h(x)}{x}$ for $|x| > \mathrm{B}$, and thus from (1.3) and the continuity of $g(x)$ and $h(x)$ that there are positive constants B_1 and B_2 such that

$$2\int_0^x [a\mathrm{G}(u) - h(u)]\,\mathrm{d}u \geq a\int_0^x \mathrm{G}(u)\,\mathrm{d}u - \mathrm{B}_1 \geq \frac{ab}{2}x^2 - \mathrm{B}_2. \tag{2.2}$$

As a result of (2.2) we have

$$\mathrm{V} \geq \frac{a^3 + ab}{2}x^2 + ay^2 + \alpha z^2 + a^2 xy - 2\,axz - 2\,yz - \mathrm{B}_2. \tag{2.3}$$

The right hand side of (2.3) can be written as $\mathrm{XCX}^{\mathrm{T}} - \mathrm{B}_2$ where $\mathrm{X} = (x, y, z)$ and

$$\mathrm{C} = \begin{pmatrix} \frac{a^3 + ab}{2} & \frac{a^2}{2} & -a \\ \frac{a^2}{2} & a & -1 \\ -a & -1 & \alpha \end{pmatrix}.$$

The eigenvalues of C will all be positive if the determinants of the principal minors are all positive. The first two are obviously positive and since $4\det \mathrm{C} = a^2(a^2 + 2\,b)\,\alpha - 2\,a(a^2 + b)$, it follows that $\det \mathrm{C} > 0$ if $\alpha > \frac{2\,(a^2 + b)}{a\,(a^2 + 2\,b)}$. Since it is assumed that α satisfies this inequality, it follows that there is $\mathrm{B}_3 > 0$ such that

$$\mathrm{V} \geq \mathrm{B}_3(x^2 + y^2 + z^2) - \mathrm{B}_2 \qquad \text{for all } x, y \text{ and } z. \tag{2.4}$$

Along a solution $(x(t), y(t), z(t))$ of Eq. (2.1) we have

$$\begin{aligned} -\dot{\mathrm{V}}_{(2.1)} &= a\left[a\frac{\mathrm{G}(x)}{x} - 2\frac{h(x)}{x}\right]x^2 + a^2 y^2 + 2\,z^2 - 2\,ayz - \\ &- 2\left[\frac{a^2}{2} + \frac{\mathrm{G}(x)}{x} - \alpha\frac{h(x)}{x}\right]xz + 2\,[az + y - \alpha z]\,p(t, x, y, z). \end{aligned}$$

If we set

$$q_1(x) = a\,\frac{G(x)}{x} - 2\,\frac{h(x)}{x} \quad \text{and} \quad q_2(x\,,\alpha) = \frac{a^2}{2} + \frac{G(x)}{x} - \alpha\,\frac{h(x)}{x}$$

then

$$-\dot{V}_{(2.1)} = XDX^T + 2\,[ax + y - \alpha z]\,p \qquad \text{where} \quad X = (x\,,y\,,z)$$

and

$$D = \begin{pmatrix} aq_1 & 0 & -q_2 \\ 0 & a^2 & -a \\ -q_2 & -a & 2 \end{pmatrix}.$$

It follows from (1.4) and (1.6) that

$$\begin{aligned} -\dot{V}_{(2.1)} &\geq XDX^T - 2\,B_0\,|ax + y - \alpha z| - \\ &- 2\,e_2(t)\,|\,ax + y - \alpha x\,|\,(x^2 + y^2 + z^2)^{1/2} \geq \\ &\geq XDX^T - 2\,B_0\,|ax + y - \alpha z| - B_4\,e_2(t)\,(x^2 + y^2 + z^2) \end{aligned}$$

where $B_4 = \max(2\,a\,,2\,\alpha\,,2)$.

Since $\det D = a^2(aq_1 - q_2^2) = a^2\,Q(x\,,\alpha)$, the eigenvalues of D will be bounded below by a positive constant B_5 if there are positive constants B_6 and B_7 such that $q_1(x) \geq B_6$ and $Q(x\,,\alpha) \geq B_7$. It follows from (1.5) that each of these inequalities is satisfied for $|x| > B$, and thus that there is $B_5 > 0$ such that

$$-\dot{V}_{(2.1)} \geq B_5\,(x^2 + y^2 + z^2) - 2\,B_0\,|ax + y - \alpha z| - B_4\,e_2(t)\,(x^2 + y^2 + z^2).$$

If $0 \leq e_2(t) \leq \varepsilon < \frac{B_5}{B_4}$, then there is $B_8 > 0$ such that

$$-\dot{V}_{(2.1)} \geq 1 \qquad \text{for} \quad x^2 + y^2 + z^2 \geq B_8. \tag{2.5}$$

Theorem 1 follows from (2.4) and (2.5), see e.g. [11, p. 11 and p. 38].

3. PROOF OF THEOREM 2

Define $E(t)$ by $E(t) = \int_0^t e_1(s)\,ds$, then it follows from (1.4) and (1.8) that $E(t)$ is monotonic increasing and that there is a positive constant E_0 such that $0 \leq E(t) \leq E_0\,t \geq 0$. Let V be the function defined in the proof of Theorem 1 and define $W = W(t\,,x\,,y\,,z)$ by

$$W = [V(x\,,y\,,z) + k]\exp(-2\,E(t))$$

where k is a positive constant to be determined later in the proof.

Proceeding as in the proof of Theorem 1, but using (1.7) and (1.8), we can find a positive constant B_1 such that

$$B_1 (x^2 + y^2 + z^2) \leq V(x, y, z) \tag{3.1}$$

for all x, y and z if $\alpha > \frac{2(a^2+b)}{a(a^2+2b)}$. It follows that

$$\begin{aligned} B_1 \exp(-2 E_0)(x^2 + y^2 + z^2) + k \exp(-2 E_0) \leq \\ \leq W(t, x, y, z) \leq V(x, y, z) + k \end{aligned} \tag{3.2}$$

for $t \geq 0$ and all x, y and z.

Again proceeding as in the proof of Theorem 1, it follows from (1.7) and (1.8) that there is $B_2 > 0$ such that along any solution $(x(t), y(y), z(t))$ of (2.1)

$$-\dot{V}_{(2.1)} \geq B_2 (x^2 + y^2 + z^2) - B_3 e_2(t)(x^2 + y^2 + z^2) - 2 e_1(t) |ax + y - \alpha z|$$

where $B_3 = \max(2a, 2\alpha, 2)$. If $0 \leq e_2(t) \leq \varepsilon < \frac{B_2}{B_3}$ and $B_4 = B_2 - \varepsilon B_3$ we have

$$\dot{V}_{(2.1)} \leq -B_4 (x^2 + y^2 + z^2) + 2 e_1(t) |ax + y - \alpha z|$$

for $t \geq 0$ and all x, y and z.

Along a soluton $(x(t), y(t), z(t))$ of Eq. (2.1) we have

$$\begin{aligned} \dot{W}_{(2.1)} = -2 e_1(t)[V + k] \exp(-E(t)) + \dot{V}_{(2.1)} \exp(-E(t)) \leq \\ \leq \exp(-E(t)) \{-B_4 (x^2 + y^2 + z^2) - 2 e_1(t)[V + k - |ax + y - \alpha z|]\}. \end{aligned}$$

It follows from (3.1) that if we set $k = \frac{a^2 + \alpha^2 + 1}{B_1}$, then

$$\dot{W}_{(2.1)} \leq -B_4 \exp(-E_0)(x^2 + y^2 + z^2) \tag{3.3}$$

for $t \geq 0$ and all x, y and z.

We note that as a result of (3.2) and (3.3), all solutions of Eq. (2.1) are bounded, and Theorem 2 follows from Yoshizawa [11, p. 61] noting that $G(0) = 0$, $h(0) = 0$ and if $(x(t), y(t), z(t))$ is a solution of Eq. (2.1), then there is $K \geq 1$ such that $x^2(t) + y^2(t) + z^2(t) \leq K$ for $t \geq 0$ and along this solution we have

$$\int_0^\infty |p(t, x(t), y(t), z(t))| \, dt \leq \int_0^\infty e_1(t) + e_2(t)(x^2(t) + y^2(t) + z^2(t)) \, dt \leq$$

$$\leq K \int_0^\infty e_1(t) + e_2(t) \, dt < \infty.$$

4. PROOF OF THEOREM 3

If we let $h(x) = \frac{a}{d} G(x) - f(x)$, then

$$Q(x, \alpha) = a\left[a \frac{G(x)}{x} - \frac{2aG(x)}{dx} + 2\frac{f(x)}{x}\right] - \\ - \left[\frac{a^2}{2} + \frac{G(x)}{x} - \frac{\alpha a G(x)}{dx} + \alpha \frac{f(x)}{x}\right]^2,$$

and setting $\alpha = \frac{d}{a}$, $d > 2$, we have

$$Q(x, \alpha) = a\left[\frac{a(d-2)G(x)}{dx} + 2\frac{f(x)}{x}\right] - \left[\frac{a^2}{2} + \frac{df(x)}{ax}\right]^2$$

where $\alpha = \frac{d}{a} > \frac{2}{a} > \frac{2(a^2+b)}{a(a^2+2b)}$. To satisfy (1.5) or (1.7) there must be $\delta > 0$ such that

$$a\left[\frac{a(d-2)G(x)}{dx} + 2\frac{f(x)}{x}\right] - \left[\frac{a^2}{2} + \frac{df(x)}{ax}\right]^2 - \delta \geq 0$$

for $|x| > B$. This inequality can be rewritten as

$$\frac{a^4}{d^4}\left[d(d-2)\frac{G(x)}{x} - a^2(d-1) - \frac{\delta d^4}{a^4}\right] \geq \left[\frac{f(x)}{x} + \frac{a^3(d-2)}{2d^2}\right]^2.$$

Setting $\varepsilon = \frac{\delta d^4}{a^4}$ and simplifying we obtain the inequality expressed in (1.10).

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