Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Niels M. Christensen, Aldo Ferrari, Giacomo Monti Bragadin

On a theorem of Malgrange for finitely differentiable functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.6, p. 835–839.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_6_835_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 29 giugno 1974

Presiede il Presidente della Classe BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On a theorem of Malgrange for finitely differentiable functions* (*). Nota di NIELS M. CHRISTENSEN, ALDO FERRARI e GIACOMO MONTI BRAGADIN, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Vengono stabiliti risultati generali su sistemi di generatori di ideali di funzioni finitamente differenziabili che soddisfano la condizione A_3 [2], la quale può sovente sostituire la condizione di chiusura. Si ottiene così fra l'altro che la $\mathscr{D}^{N+1}$–algebra $\mathscr{D}^N$ non è numerabilmente generabile.

In the theory of differentiable functions the closed ideals in the topology of compact convergence play an important part. For finitely differentiable functions the condition "closed" can often be replaced by a weaker condition [3, p. 334], which is generally known as A_3, [2]. In [1], Malgrange proved that a (finite) set of analytic functions generates a closed ideal in the ring of infinitely differentiable functions. For functions of *one* variable two results are due to Spallek [4]. In the infinitely differentiable case every pointwise finitely generated closed ideal is generated by one analytic function, and in the finitely differentiable case there exists no proper pointwise finitely generated closed ideal. Thus the theorem of Malgrange does not hold for finitely differentiable functions. We show (for the exact result see below) that in the finitely differentiable case there are no finitely or denumerably generated ideals which fulfill the weaker condition A_3 (for principal ideals see [5]). An immediate consequence of this result is a remark about the structure of the $\mathscr{D}^{N+1}$–algebra $\mathscr{D}^N$.

(*) Gli autori italiani hanno usufruito, durante la stesura del lavoro, di una borsa C.N.R., comitato per le Scienze Matematiche, per l'estero.

(**) Nella seduta del 29 giugno 1974.

We denote by $\mathscr{D}^N$ the ring of germs of N-times continuously differentiable real or complex valued functions in the origin of $\mathbf{R}^n = \{(x_1, \cdots, x_n)\}$, n arbitrary. Thus for any ideal $\mathscr{I}$ in $\mathscr{D}^N$, $1 \leq N < \infty$, $\mathscr{I}$ fulfills condition A_3 iff

$$(\mathscr{I} \cdot \mathscr{D}^{N-1}) \cap \mathscr{D}^N = \mathscr{I}.$$

For a set S of generators of the proper ideal $\mathscr{I} \subset \mathscr{D}^N$ we denote by $S_1 := S \cap (\mathscr{D}^N \setminus \mathscr{D}^{N+1})$ the set of those generators, which are not of class C^{N+1}. Then we have the following theorem.

THEOREM. *If* S_1 *is finite or denumerably infinite, then the ideal* $\mathscr{I}$ *does not fulfill condition* A_3 *and hence cannot be closed.*

Consequently no set of C^{N+1}-functions or analytic functions generates a closed ideal.

COROLLARY. *The* $\mathscr{D}^{N+1}$*-algebra* $\mathscr{D}^N$ *is not denumerably generated.*

Proof of Corollary. Assume that there exists denumerably many functions $g_i \in \mathscr{D}^N$, $i \in \mathbf{N}$, generating the $\mathscr{D}^{N+1}$-modul $\mathscr{D}^N$. Then

$$\mathscr{D}^N = \sum_{i \in \mathbf{N}} g_i \, \mathscr{D}^{N+1} = \sum_{i \in \mathbf{N}} (g_i - g_i(0)) \cdot \mathscr{D}^{N+1} + 1 \cdot \mathscr{D}^{N+1}$$

so that

$$\mathfrak{m}_N = \sum_{i \in \mathbf{N}} (g_i - g_i(0)) \cdot \mathscr{D}^{N+1} + \mathfrak{m}_{N+1},$$

where $\mathfrak{m}_N$ denotes the maximal ideal in $\mathscr{D}^N$. This contradicts the theorem, because $\mathfrak{m}_{N+1}$ is contained in $\mathscr{D}^{N+1}$ and $\mathfrak{m}_N$ fulfills A_3. Since $\mathscr{D}^N$ is not denumerably generated as $\mathscr{D}^{N+1}$-modul it is not a denumerably generated $\mathscr{D}^{N+1}$-algebra.

For proving the theorem we need a technical lemma.

LEMMA. *Let* $f : U \to \mathbf{R}(\mathbf{C})$, U *open in* $\mathbf{R}^n$, *be a* C^N*-function with* $f(0) = 0$ *and* (x^ν), $x^\nu \in U$, *be a sequence converging to* 0 *with* $f(x^\nu) \neq 0$ *and* $x^\nu \neq x^\mu$ *for* $\nu \neq \mu$. *Let* (α^ν) *be any sequence of positive numbers converging to* 0.

Then there exists a function $\gamma : U \to \mathbf{R}(\mathbf{C})$ *which satisfies the following conditions*:

$$(1) \qquad \gamma \textit{ is of class } C^{N-1},$$

$$(2) \qquad \gamma \cdot f \textit{ is of class } C^N,$$

$$(3) \qquad \frac{1}{|x^\nu|} \cdot \frac{\partial^j \gamma}{\partial x_1^j}(x^\nu) \textit{ converges to } 0 \textit{ for } 0 \leq j < N,$$

$$(4) \qquad f(x^\nu) \cdot \frac{\partial^N \gamma}{\partial x_1^N}(x^\nu) = \alpha^\nu.$$

Proof. We first construct a function $\gamma : U \to \mathbf{R}(\mathbf{C})$ with the properties:

(5) for all $j = (j_1, \cdots, j_n)$ with $|j| \leq N$ and $j_1 \neq N$ and all $x \in U$ the derivatives $\dfrac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} \gamma(x)$ exist, are continuous and equal to 0 in the origin,

(6) for all $x \in U \setminus \{0\}$ the derivative $\frac{\partial^N}{\partial x_1^N} \gamma(x)$ exists and is continuous in $U \setminus \{0\}$,

(7) $f \cdot \frac{\partial^N \gamma}{\partial x_1^N}$ is continuous in the origin,

and with the properties (3) and (4).

Construction of γ: Let $\xi : \mathbf{R}^n \to [0, 1]$ be of class C^∞ with $\xi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\xi(x) = 0$ for $|x| \geq 1$. Choose a sequence of numbers $\varepsilon^\nu \leq \frac{1}{2} |x^\nu|$ satisfying $|f(x)| \leq 2 |f(x^\nu)|$ for all $x \in U_{\varepsilon^\nu}(x^\nu)$ and $U_{\varepsilon^\nu}(x^\nu) \cap U_{\varepsilon^\mu}(x^\mu) = \varnothing$ for $\nu \neq \mu$, where $U_{\varepsilon^\nu}(x^\nu)$ denotes the open disk around x^ν with radius ε^ν. Next define the sequence (δ^ν) by $\delta^\nu := \min \{2^{-\nu} \cdot (\varepsilon^\nu)^N \cdot |f(x^\nu)| \cdot |x^\nu| ; \varepsilon^\nu\}$, $(\varepsilon^\nu)^N$ being the N-th power of ε^ν. We construct γ by N-times integration

$$\gamma(x_1, \cdots, x_n) :=$$

$$= \sum_{\nu=1}^{\infty} \alpha^\nu \frac{1}{f(x^\nu)} \cdot \int_0^{x_1} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-1}} d\tau_N \, \xi\left(\frac{\tau_N - x_1^\nu}{\delta^\nu}, \frac{x_2 - x_2^\nu}{\varepsilon^\nu}, \cdots, \frac{x_n - x_n^\nu}{\varepsilon^\nu}\right).$$

Taking account of the definitions of ε^ν and δ^ν, properties (3) to (7) are easily seen to be verified.

Finally it is clear that (1) and (2) follows from (5), (6) and (7) and hence γ satisfies the lemma.

Proof of the theorem. We will show the following sharper result. For any function $\bar{f}$, $0 \neq \bar{f} \in S$, there exists a C^{N-1}-function γ with $\gamma\bar{f}$ being of class C^N and $\gamma\bar{f} \notin S \cdot \mathscr{D}^N$.

Changing the coordinates suitably we have a sequence (x^ν) converging towards 0 with $\bar{f}(x^\nu) \neq 0$ and $x^\nu \in K := \{x \mid |x| \leq \sqrt{n}\,|x_1|\} \subset \mathbf{R}^n$. For the sequel x will denote an element of this cone, and the *i-th* derivative of any function f in the direction of x_1 will be denoted by $f^{(i)}$.

Every function $f \in S$ has a unique representation

$$f = p + r_f \tag{8}$$

with p being a polynomial of degree less or equal N and r_f a N-flat function, i.e. a function with zero-polynomial as N-th Taylor polynomial in the origin. We call r_f the *remainder of f*.

For any C^N-function g and $x_1 \neq 0$ we have

$$[(g \cdot f)^{(N)}(x) - (g \cdot f)^{(N)}(0)] \cdot \frac{1}{x_1} =$$

$$= \left[\sum_{i=0}^{N} b_i \, g^{(N-i)}(x) \cdot p^{(i)}(x) - b_i \, g^{(N-i)}(x) \cdot r_f^{(i)}(x) - \sum_{i=0}^{N} b_i \, g^{(N-i)}(0) \cdot p^{(i)}(0)\right] \cdot \frac{1}{x_1},$$

where b_i are certain binomial coefficients.

For x in a neighbourhood of o and in K,

$$[g^{(N-i)}(x) \cdot p^{(i)}(x) - g^{(N-i)}(0) \cdot p^{(i)}(0)] \cdot \frac{1}{x_1}$$

is bounded

i) for $i \neq 0$ because $g^{(N-i)} \cdot p^{(i)}$ is a C^1–function, and

ii) for $i = 0$ because $p^{(0)}$ is a polynomial vanishing in the origin (take account of the fact that $x \in K$).

Since r_f is N–flat,

$$g^{(N-i)}(x) \cdot r_f^{(i)}(x) \cdot \frac{1}{x_1}$$

is bounded in a neighbourhood of o intersected with K for $i \neq N$. So for $x \in K$, $|x|$ small,

$$\left| [(g \cdot f)^{(N)}(x) - (g \cdot f)^{(N)}(0)] \cdot \frac{1}{x_1} \right| \leq c_1 + c_2 \left| \frac{r_f^{(N)}(x)}{x_1} \right| \tag{9}$$

with suitable $c_1, c_2 \in \mathbf{R}$ (depending on f and g).

Using the lemma we shall now construct (by choosing a suitable sequence (α^ν)) a C^{N-1}–function γ, such that γf is of class C^N and has no representation

$$\gamma f = \sum_{k=1}^{s} g^k f^k \quad , \quad g^k \ C^N\text{–functions}, \ f^k \in S. \tag{10}$$

Since $S_1 = S \cap (\mathcal{D}^N \setminus \mathcal{D}^{N+1})$ is finite or denumerable, we have an enumeration

$$S_1 =: \{f_j \mid j \in I\} \quad \text{with } I \subset \mathbf{N}. \tag{11}$$

We denote by r_j the remainders of the functions f_j in a representation of type (8). There exists a sequence (β^ν) of real numbers converging towards o and a monotonic sequence $(\eta(\nu))$ of positive integers tending to infinity such that

$$\beta^\nu \geq \sqrt{|x^\mu|} \quad \text{for all } \mu \geq \nu \tag{12}$$

$$\beta^\nu \geq |r_j^{(N)}(x^\mu)| \quad \text{for all } \mu \geq \nu \text{ and all } j \leq \eta(\nu). \tag{13}$$

We then define the sequence (α^ν) by

$$\alpha^\nu := \sqrt{\beta^\nu} \tag{14}$$

and choose a function γ according to the lemma. Since γ is N-times continuously differentiable in a neighbourhood of x^ν and satisfies (3) and (4) we have with a suitable constant c_3

$$\left| [(\gamma f)^{(N)}(x^\nu) - (\gamma f)^{(N)}(0)] \frac{1}{x_1^\nu} \right| = \tag{15}$$

$$= \left| \sum_{i=0}^{N} b_i \gamma^{(i)}(x^\nu) \cdot \bar{f}^{(N-i)}(x^\nu) \frac{1}{x_1^\nu} \right| \geq -c_3 + \frac{\alpha^\nu}{|x_1^\nu|}$$

for all ν sufficiently large.

We assume that $\gamma\bar{f}$ satisfies an equation (10) and have

$$-c_3+\frac{\alpha^\nu}{|x_1^\nu|}\leq\left|[(\gamma\bar{f})^{(N)}(x^\nu)-(\gamma\bar{f})^{(N)}(0)]\frac{1}{x_1^\nu}\right| \qquad \text{by (15)}$$

$$=\left|\left[\sum_{k=1}^{s}(g^k f^k)^{(N)}(x^\nu)-(g^k f^k)^{(N)}(0)\right]\frac{1}{x_1^\nu}\right| \qquad \text{by (10)}$$

$$\leq c_4+c_5\sum_{k=1}^{s}\left|\frac{r_{fk}^{(N)}(x^\nu)}{x_1^\nu}\right|; \qquad c_4, c_5\in\mathbf{R} \qquad \text{by (9).}$$

For ν sufficiently large, such that, in particular, $\eta(\nu)$ exceeds the maximal index in the enumeration (11) of those f^k which belong to S_1, we have:

i) $|r_{fk}^{(N)}(x^\nu)|\leq\beta^\nu$ for those f^k belonging to $S\setminus S_1$, by (12),

and

ii) $|r_{fk}^{(N)}(x^\nu)|\leq\beta^\nu$ for those f^k belonging to S_1, by (13).

Consequently, for large ν and suitable constants c and c' we obtain

$$\frac{\alpha^\nu}{|x_1^\nu|}\leq c+c'\frac{\beta^\nu}{|x_1^\nu|},$$

which contradicts our definition (14). This proves the theorem.

REFERENCES

[1] MALGRANGE B. (1966) – *Ideals of differentiable functions*, Oxford University Press.
[2] SPALLEK K. (1969) – *Differenzierbare Räume*, «Math. Ann.», *180*, 269–296.
[3] SPALLEK K. (1972) – *Beispiele zur lokalen Theorie der differenzierbaren Räume*, «Math. Ann.», *195*, 332–337.
[4] SPALLEK K. (1972) – *Abgeschlossene Garben differenzierbarer Funktionen*, «Man. Math.», *6*, 147–175.
[5] MONTI-BRAGADIN G. (1973) – *Some topics on differentiable spaces*, «Rend. Mat.», *4*, ser. 6, 295–305.