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Angelo Di Tommaso, Antonio Tralli

Reliability tests for structures under general loading processes. Nota II

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Meccanica. — *Reliability tests for structures under general loading processes.* Nota II di ANGELO DI TOMMASO e ANTONIO TRALLI, presentata [(*)] dal Corrisp. E. GIANGRECO.

RIASSUNTO. — Vengono istituiti, mediante la programmazione matematica, i tests di affidabilità relativi ai modelli matematici costruiti nella Nota I.

6. RELIABILITY TESTS

6.1. *General formulation.*

The general formulation of a *Reliability test*, as it appears from the above considerations, consists in verifying if for the sets $\mathscr{T}^*$ and $\mathscr{E}^*$, defined by (37) and (38), the statement:

$$\mathscr{T}^* \subseteq \mathscr{E}^* \tag{41}$$

is *true* or *false*.

In alternative form, if for the sets $\mathscr{T}$ and $\mathscr{E}$, defined by (35) and (40), the statement:

$$\mathscr{T} \subseteq \mathscr{E} \tag{42}$$

is *true* or *false*.

In fact it appears obvious that:

$$\mathscr{T}^* \subseteq \mathscr{E}^* \Longleftrightarrow \mathscr{T} \subseteq \mathscr{E} . \tag{43}$$

This formulation can contain also a *shake-down reliability test* if a suitable dislocation vector $\mathbf{I}^\Delta$ which represents a particular $\mathbf{I}$ whose elements are all zero except for the dislocation subvector Δ:

$$\tilde{\mathbf{I}}^\Delta = [\tilde{\mathbf{o}} \mid \tilde{\mathbf{o}} \mid \tilde{\Delta} \mid \tilde{\mathbf{o}}] \tag{44}$$

is introduced.

Then the first shake-down theorem or *Melan's* theorem can be so enunciated:

> the structural model shakes-down if and only if
>
> $$\exists \tilde{\mathbf{I}}^\Delta = [\tilde{\mathbf{o}} \mid \tilde{\mathbf{o}} \mid \tilde{\Delta} \mid \tilde{\mathbf{o}}] : \mathscr{T}^* \{\mathbf{I}^* \mid \mathbf{I}^* = \mathbf{A}^{-1} (\mathbf{I}^\Delta + \mathbf{I}) ,\ \mathrm{I} \in \mathscr{T}\} \subseteq \mathscr{E}^{*y}$$

where $\mathscr{E}^{*y}$ is the domain $\mathscr{E}^*$ without constraints on displacements and reactions.

(*) Nella seduta del 20 aprile 1974.

6.2. *Local test.*

In many cases it is required only the evaluation of the maxima and minima of some of $\mathbf{I}^*$ components; i.e. the maximum and minimum value of displacements or internal stresses or reactions of constraints only in certain prefixed points or elements as is usual in a practical approach. This is equivalent to assigning a very simple $\mathscr{E}^*$ domain i.e. a domain with few faces.

Therefore it may be convenient to proceed in the following way: let us consider matrix (29); in this matrix each element is an *influence coefficient* of the external actions; the search for maximum and minimum of the single components (4) of $\mathbf{I}^*$ is a classical problem of *linear programming* [11], [12].

a) *Maximum in nodal displacements*

$$\max\,[\boldsymbol{u}_\alpha]_i = [(\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \mid \mathbf{o} \mid (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \tilde{\mathbf{B}}_\alpha \mathbf{D} \mid -(\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)\, \tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\beta]_i \cdot \begin{bmatrix} \mathbf{F}_\alpha \\ \hdashline \mathbf{o} \\ \hdashline \varDelta \\ \hdashline \boldsymbol{u}^0_\beta \end{bmatrix} \tag{46.a}$$

subject to constraints:

$$\mathbf{V}\,\mathbf{I} \leq \mathbf{L}\,. \tag{46.b}$$

b) *Maximum in internal stresses*

$$\begin{aligned} \max\,[\mathbf{Q}]_i = [&\mathbf{D}\mathbf{B}_\alpha (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \mid \mathbf{o} \mid \mathbf{D}\mathbf{B}_\alpha (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \tilde{\mathbf{B}}_\alpha \mathbf{D} - \mathbf{D} \mid - \\ &- \mathbf{D}\mathbf{B}_\alpha (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\beta + \mathbf{D}\mathbf{B}_\beta]_i \cdot \begin{bmatrix} \mathbf{F}_\alpha \\ \hdashline \mathbf{o} \\ \hdashline \varDelta \\ \hdashline \boldsymbol{u}^0_\beta \end{bmatrix} \end{aligned} \tag{47.a}$$

subject to constraints:

$$\mathbf{V}\mathbf{I} \leq \mathbf{L}\,. \tag{47.b}$$

c) *Maximum in constraint reactions*

$$\begin{aligned} \max\,[-\mathbf{F}_\beta]_i = [&-\tilde{\mathbf{B}}_\beta \mathbf{D}\mathbf{B}_\alpha (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \mid \mathbf{I} \mid -\tilde{\mathbf{B}}_\beta \mathbf{D}\mathbf{B}_\alpha (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \tilde{\mathbf{B}}_\alpha \mathbf{D} + \\ &+ \tilde{\mathbf{B}}_\beta \mathbf{D} \mid \tilde{\mathbf{B}}_\beta \mathbf{D}\mathbf{B}_\alpha (\tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\alpha)^{-1} \tilde{\mathbf{B}}_\alpha \mathbf{D}\mathbf{B}_\beta - \tilde{\mathbf{B}}_\beta \mathbf{D}\mathbf{B}_\beta]_i \cdot \begin{bmatrix} \mathbf{F}_\alpha \\ \hdashline \mathbf{o}_\beta \\ \hdashline \varDelta \\ \hdashline \boldsymbol{u}^0_\beta \end{bmatrix} \end{aligned} \tag{48.a}$$

(4) In what follows $[\boldsymbol{v}]_i$ represents the i–component of the vector $\boldsymbol{v}$, while it represents the i–row if $\mathbf{V}$ is a matrix.

subject to constraints:

$$\mathbf{VI} \leq \mathbf{L} . \tag{48.b}$$

This can be considered as a new approach of the usual *influence function method*; it is a convenient and powerful approach when the number of conditions is large and connected.

We note that the search for maximum in internal stress component $[\mathbf{Q}]_i$ can satisfy us obviously only in the case of pin-joined trusses where uniaxial stress occurs, but in general cases we have problems with no-uniaxial stress state. Then we need to work with a yield criterion for triaxial stress state; in many cases Von Mises Criterion is suitable. The second invariant of stress tensor can be represented by quadratic form in each constant-stress element j:

$$\sigma_e^j = \tilde{\sigma}^j \mathbf{H} \sigma^j \tag{49}$$

where $\mathbf{H}$ is the hessian matrix of the quadratic form and σ^j is the stress vector (3) of the j element.

From (14) we have:

$$\sigma_e^j = \frac{1}{(V^j)^2} \tilde{\mathbf{Q}}^j (\mathbf{T}^j \mathbf{H} \tilde{\mathbf{T}}^j) \mathbf{Q}^j \tag{50}$$

where $\mathbf{Q}^j$ vector is defined in (12).

We are interested to find out:

$$\max \{ \sigma_e^j \mid \mathbf{VI} \leq \mathbf{L} \} . \tag{51}$$

Taking into account (26), (29), it is easy to see that it is a classical problem of *quadratic programming* [13].

An alternative procedure is to use a piecewise linear approximation of the yield surface [8]; in this case for every element the yield surface is represented by:

$$\boldsymbol{n}^j \sigma^j \leq \boldsymbol{k}^j . \tag{52}$$

Let the dimensions of vector k^j be t (i.e. the number of inequalities) then from (14), with:

$$\mathbf{N}^j = \frac{1}{V^j} \boldsymbol{n}^j \tilde{\mathbf{T}}^j$$

we have:

$$\mathbf{N}^j \mathbf{Q}^j \leq \mathbf{K}^j . \tag{53}$$

A face of polihedron is:

$$[\mathbf{N}^j]_i \mathbf{Q}^j \leq [\mathbf{K}^j]_i \qquad (i = 1 , 2 , \cdots , t) \tag{54}$$

then for every i we can have, by *linear programming*:

$$\max \{ [\mathbf{N}^j]_i \mathbf{Q}^j \mid \mathbf{VI} \leq \mathbf{L} \}$$

and we have the possibility to verify whether or not the value of objective is less than $[\mathbf{K}^j]_i$.

6.3. *Complete test.*

A complete test, in the sense of (43), can be performed only by a very hard computational work.

Because every set $\mathcal{T}^*$, $\mathcal{E}^*$, $\mathcal{T}$, $\mathcal{E}$ is represented through a certain number of linear inequalities, by a geometric point of view the problem consists in recognizing if a certain hyper-polyhedron is contained or not in another one (fig. 3).

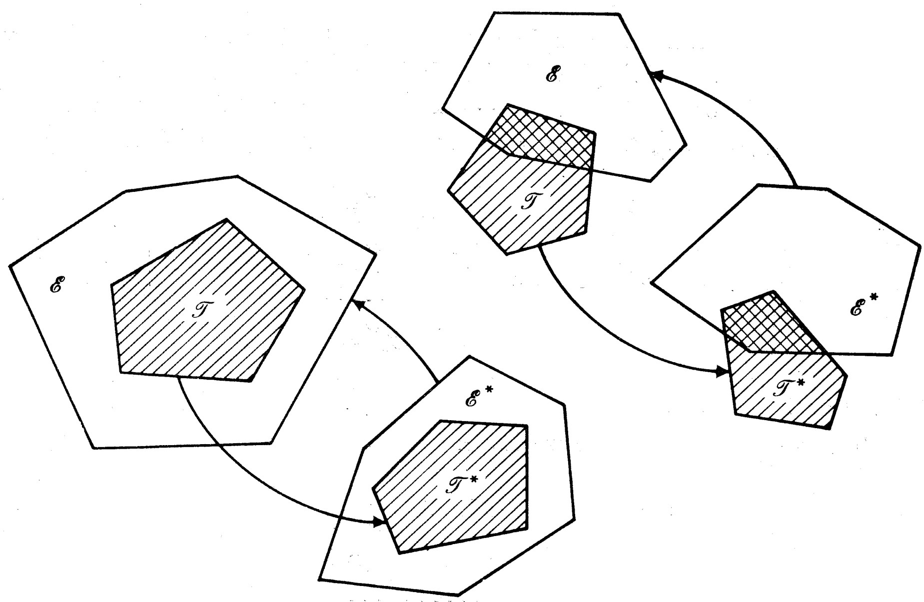


Fig. 3.

Let us examine some possible procedures for the above mentioned purpose.

The first and quite obvious procedure is to verify if all the vertexes of the first hyper-polyhedron belong to the second one; this leads to a very heavy computational work.

We suggest a second, more convenient, procedure that starts from the following considerations.

It is obvious that:

$$\mathcal{E}^* = \mathcal{E}_1^* \cap \mathcal{E}_2^* \cap \mathcal{E}_3^* \cap \cdots \cap \mathcal{E}_{n_e}^* \tag{55}$$

where every $\mathscr{E}_i^*$ is represented by a linear inequality (geometrically it is a half space); it follows that the (32) can be written:

$$\mathscr{T}^* \subseteq \mathscr{E}^* \Longleftrightarrow \mathscr{T}^* \subseteq [\mathscr{E}_1^* \cap \mathscr{E}_2^* \cap \cdots \cap \mathscr{E}_{n_e}^*] \tag{56.a}$$

then the (56.a) is verified if and only if the n_e inequalities:

$$\mathscr{T}^* \subseteq \mathscr{E}^* \Longleftrightarrow \begin{cases} \mathscr{T}^* \subseteq \mathscr{E}_1^* \\ \mathscr{T}^* \subseteq \mathscr{E}_2^* \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \mathscr{T}^* \subseteq \mathscr{E}_{n_e}^* \end{cases} . \tag{56.b}$$

are all verified.

Remembering that the complement S′ of a set S is the set of the elements which do not belong to S (that is the difference between the *Universal Set* U and S), we denote with $\mathscr{E}_1^{*\prime}, \mathscr{E}_2^{*\prime}, \cdots, \mathscr{E}_{n_e}^{*\prime}$ respectively the complement of the sets $\mathscr{E}_1^*, \mathscr{E}_2^*, \cdots, \mathscr{E}_{n_e}^*$, then we have an equivalent form of (56.b).

$$\mathscr{T}^* \subseteq \mathscr{E}^* \Longleftrightarrow \begin{cases} \mathscr{T}^* \cap \mathscr{E}_1^{*\prime} = \varnothing \\ \mathscr{T}^* \cap \mathscr{E}_2^{*\prime} = \varnothing \\ \cdots\cdots\cdots\cdots \\ \mathscr{T}^* \cap \mathscr{E}_{n_e}^{*\prime} = \varnothing \end{cases} \tag{41.b}$$

where $\varnothing$ is the *Null Set*.

With the same procedure we have:

$$\mathscr{T} \subseteq \mathscr{E} \Longleftrightarrow \begin{cases} \mathscr{T} \cap \mathscr{E}_1' = \varnothing \\ \mathscr{T} \cap \mathscr{E}_2' = \varnothing \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \mathscr{T} \cap \mathscr{E}_{n_e}' = \varnothing \end{cases} \tag{42.b}$$

where $\mathscr{E}_i'$ is the complement of the set $\mathscr{E}_i$ which is represented by a linear inequality deduced from the n_e inequalities that form the hyper-polihedron $\mathscr{E}$.

At this point for the proposed model it seems convenient to verify the (41.b) and (42.b) instead of (41) and (42).

In fact in our case $\mathscr{T}$ (ro $\mathscr{T}^*$) is defined by a set of linear inequalities while $\mathscr{E}_i'$ (or $\mathscr{E}_i^{*\prime}$) by a single linear inequality; then we can verify (42.b) [or (41.b)] in an easy way. It is possible to use the phase 1 of the simplex method [3], [4], which consists in finding a feasible solution to a L.P. problem.

In fact to verify (42.b) [or (41.b)]:

$$\mathcal{T} \subseteq \mathcal{E} \Longleftrightarrow \mathcal{T} \cap \mathcal{E}'_i = \varnothing \qquad (i = 1, \cdots, n_e),$$

it is equivalent to verify that every i system:

$$\text{(42.c)} \qquad \begin{cases} \mathbf{VI} \leq \mathbf{L} \\ [\mathbf{N}\,\mathbf{A}^{-1}\,\mathbf{I}]_i > \mathbf{M}_i \end{cases}$$

does not admit any feasible solution of any arbitrary chosen linear program. It seems convenient to remember here what is the phase 1 of simplex method. The basic method starts with a first trial solution in which all the genuine variables equal zero. But if some constraints are equations, or inequalities that are not satisfied at the select trial solution this is not feasible. Then artificial variables z_j are introduced, representing the discrepancy between the left and right hand side of the equations (the sign of these is such that the artificial variables are not negative at the trial solution). So it is possible to solve these constraints for the artificial variables making them basic. The only trouble is that the problem is changed and the solution of the new problem is also the solution of the original problem if and only if all the artificial variables z_j equal zero. Since $z_j \geq 0$ it is sufficient to make $\Sigma_j z_j = 0$ (sum of infeasibility). Therefore we have to minimize the sum of artificial variables and if this is zero there exists a genuine feasible solution of the original problem; if this does not the original problem is infeasible. This means that the original set (i.e. (42.c)) is null.

A third suitable procedure could be a generalization of above mentioned local test, i.e. we have positive test if and only if:

$$\forall i : \max\{[\mathbf{N}]_i\, \mathbf{I}^* \mid \mathbf{I}^* = \mathbf{A}^{-1}\,\mathbf{I},\ \mathbf{VI} \leq \mathbf{L}\} \leq [\mathbf{K}]_i \qquad (i = 1, 2, \cdots).$$

This way probably needs more computational time than the previous one (see fig. 4).

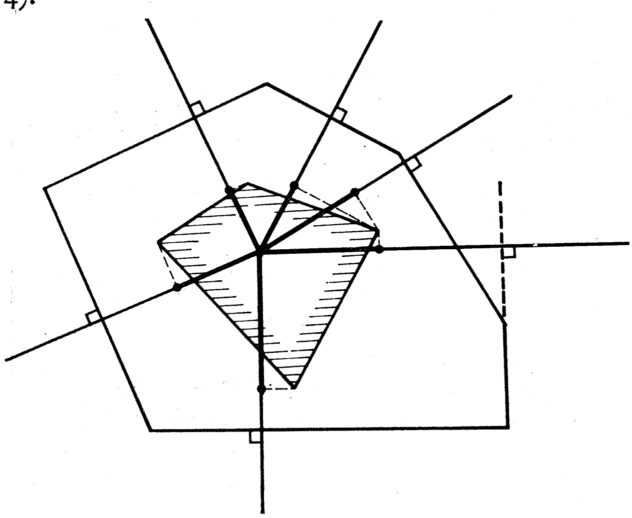


Fig. 4.

7. Numerical examples

The truss of fig. 5 allows some easy considerations on the previous theory.

The truss is supposed to be loaded only in upper nodes by vertical forces such that their whole intensity is always equal to $2P = 12{,}600$ kg and each one is less or equal to $3/2$ P. Then the diagonal bars are subjected to a

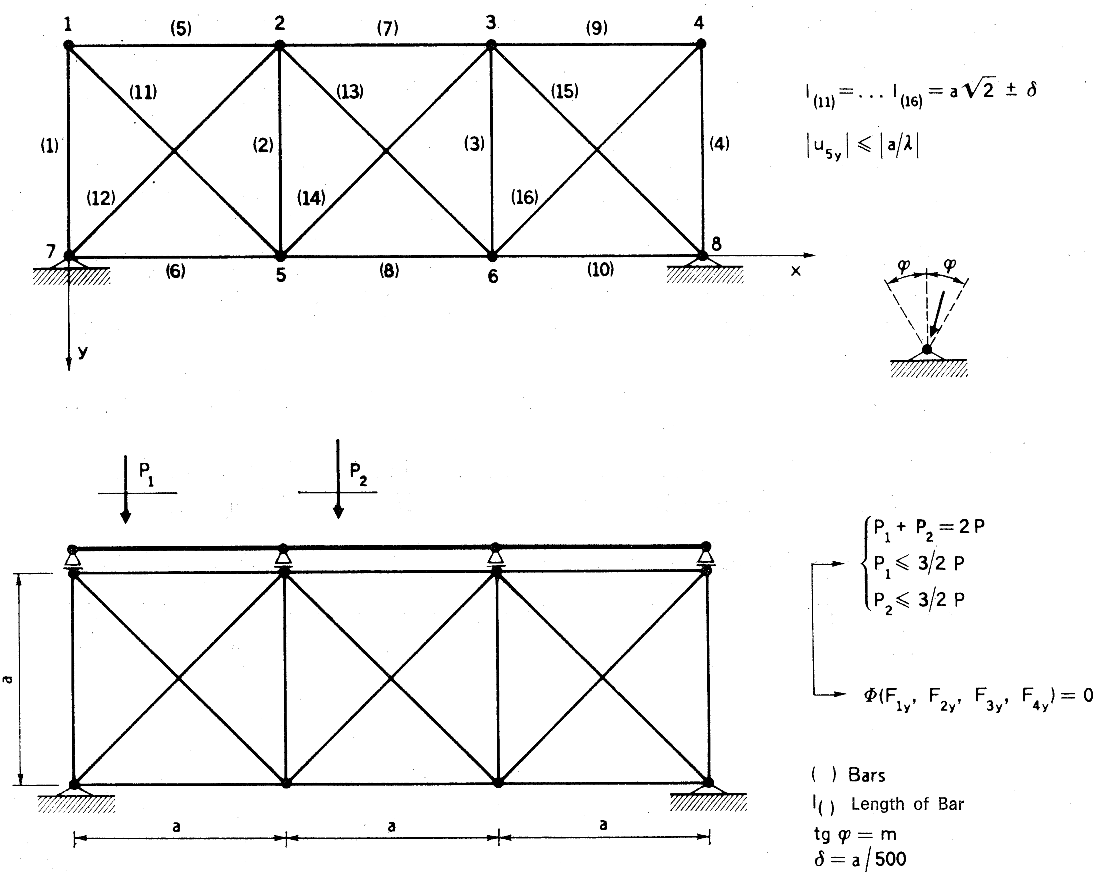


Fig. 5. – Set $\mathscr{T}$ of Load Condition Vectors for a Pin–Joined Truss.

possible dislocation in length $\pm a/500$ where $a = 6.00$ m is the length of the side of square meshes. This condition allows us to build the matrix **V** in fig. 6.

In order to bound some displacements, internal forces or constraints, we define the matrices **N** and **M** shown in fig. 7.

$$\boxed{\underline{V}\ \underline{I} \leqslant \underline{L}}$$

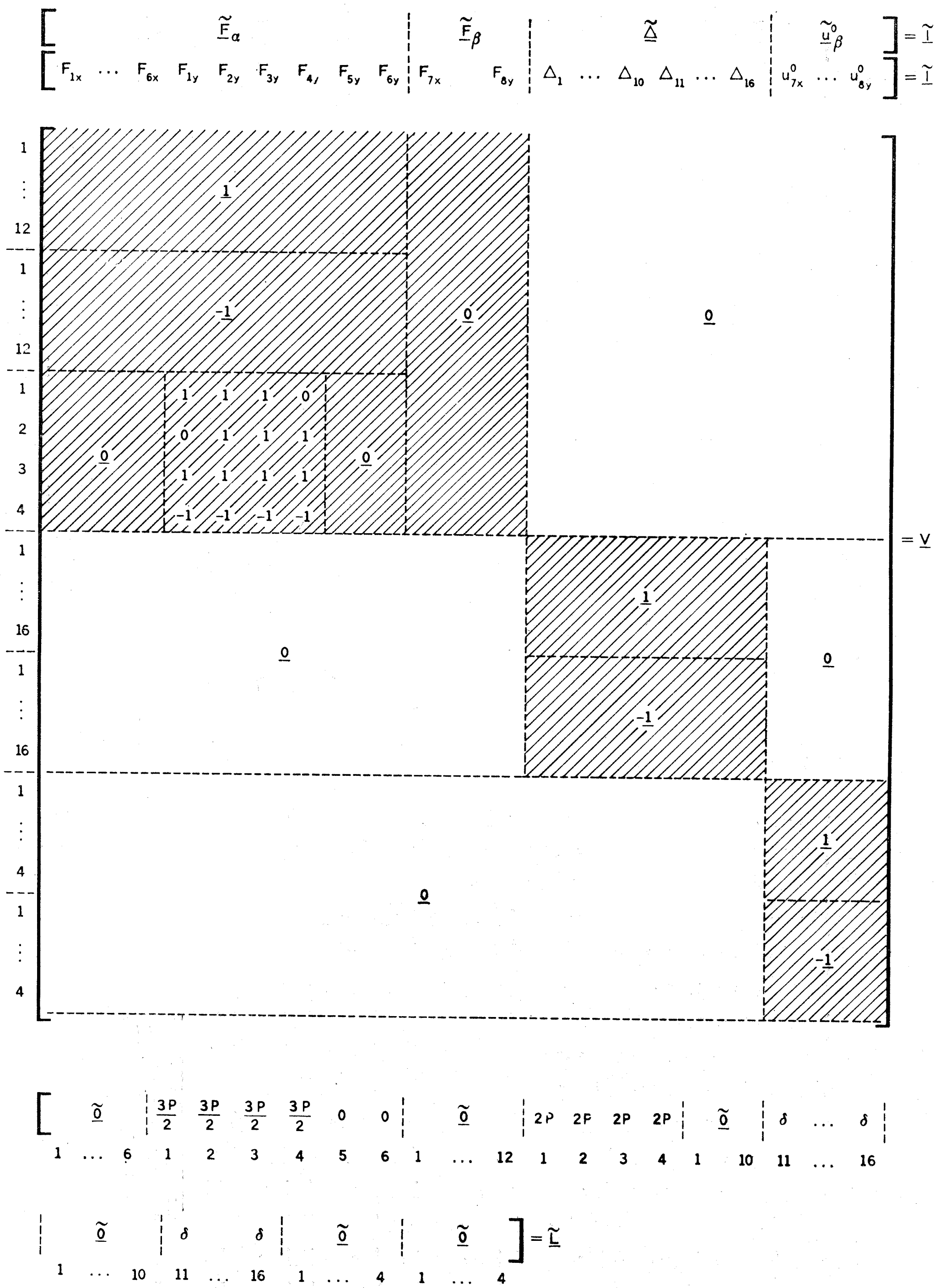


Fig. 6. – Mathematical Model of Set $\mathscr{T}$ for the Pin–Joined Truss of Fig. 5.

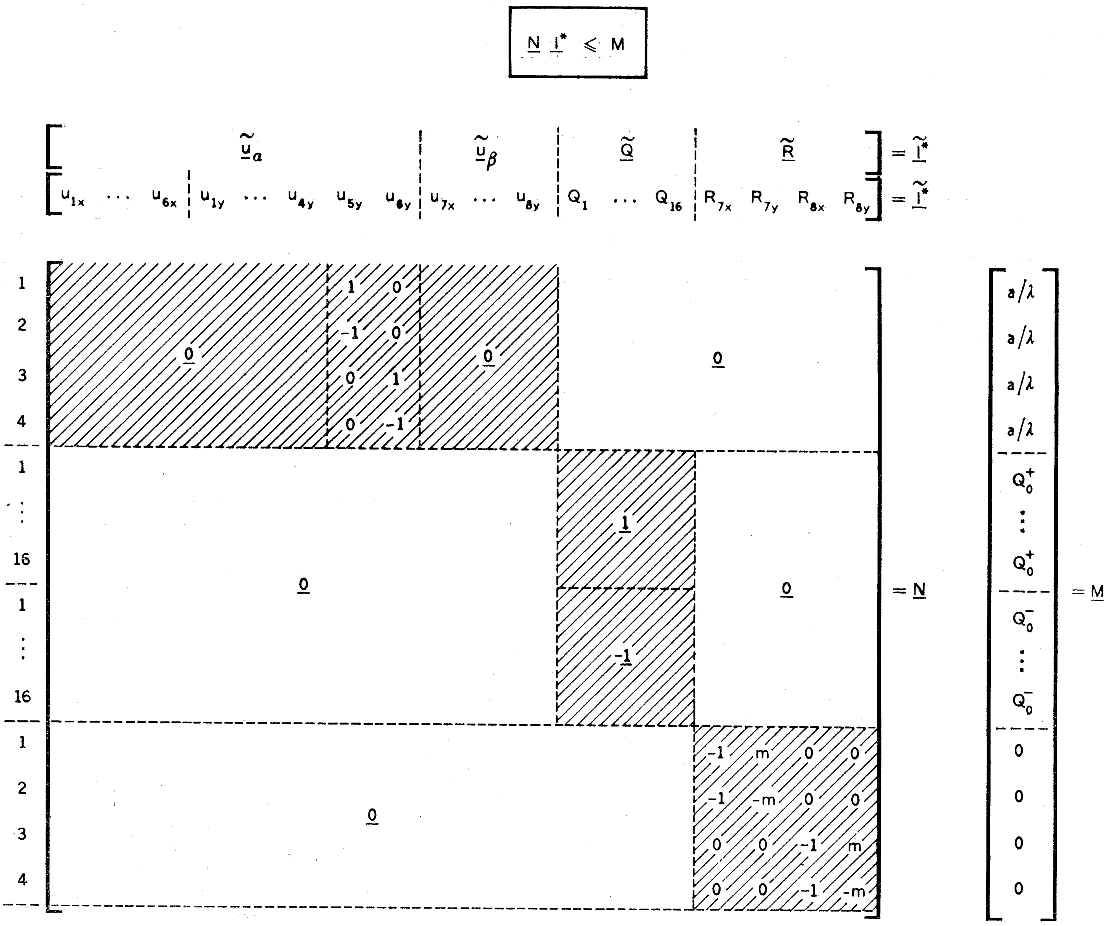
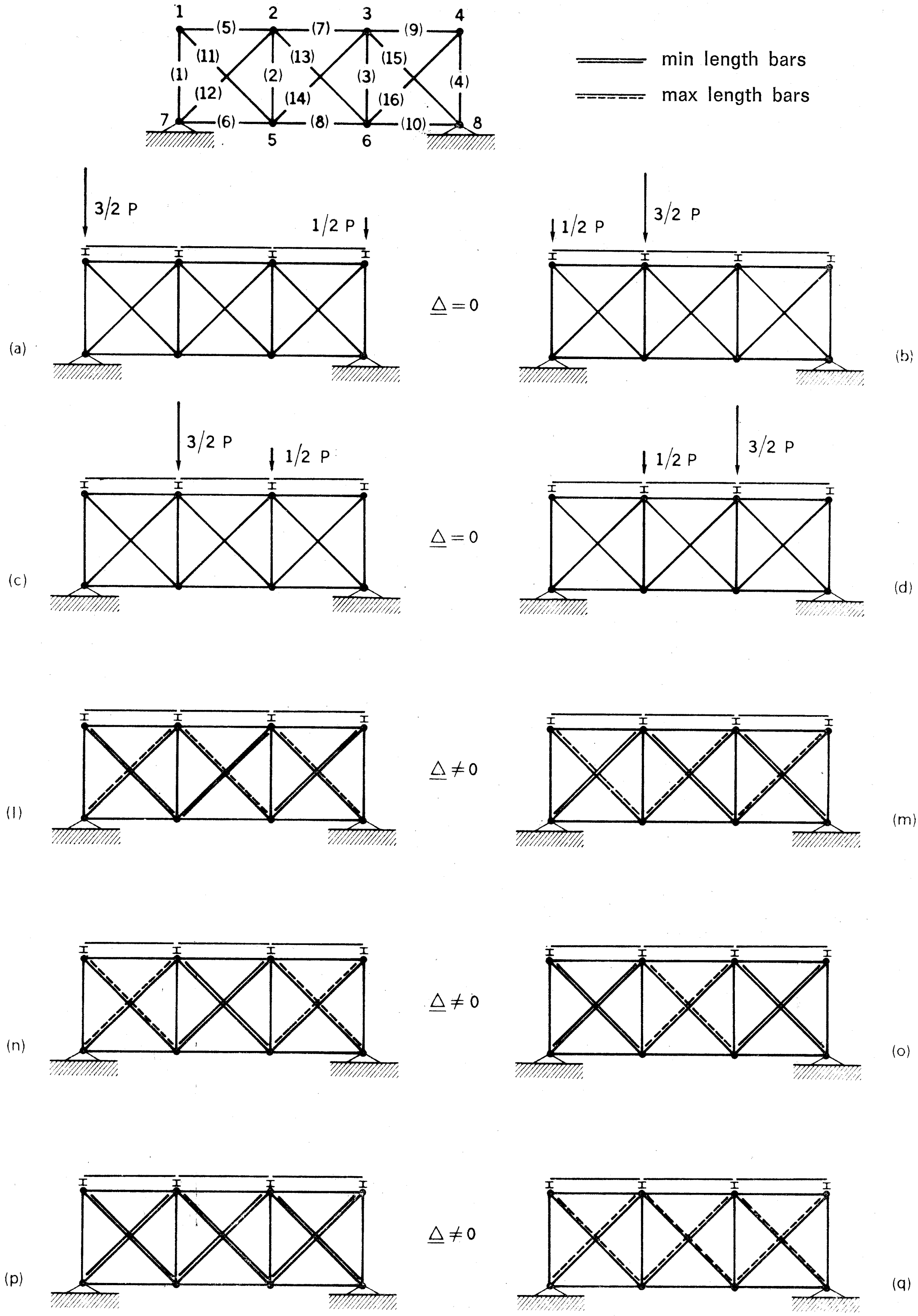


Fig. 7. – Mathematical Model of Set $\mathscr{E}^*$ for the Pin–Joined Truss of Fig. 5.

We performed some *local tests* in the sense explained previously (bar section was considered $A = 40.80 \text{ cm}^2$) and the results are listed below; (for the load conditions see fig. 8):

min $u_{5y} = 2.20 \cdot 10^{-2}$ cm	[$\Delta = 0$; load condition (*a*)]
min $u_{5y} = -2.25$ cm	[$\Delta \neq 0$; load condition (*a*), (*l*)]
max $u_{5y} = 9.02 \cdot 10^{-2}$ cm	[$\Delta = 0$; load condition (*b*)]
max $u_{5y} = 2.35$ cm	[$\Delta \neq 0$; load condition (*b*), (*m*)]
min $Q_8 = -4.80 \cdot 10^2$ kg	[$\Delta = 0$; load condition (*a*)]
min $Q_8 = -4.07 \cdot 10^4$ kg	[$\Delta \neq 0$; load condition (*a*), (*n*)]
max $Q_8 = 2.07 \cdot 10^3$ kg	[$\Delta = 0$; load condition (*c*)]
max $Q_8 = 4.23 \cdot 10^4$ kg	[$\Delta \neq 0$; load condition (*c*), (*o*)]
min $R_{8x} = 2.80 \cdot 10^2$ kg	[$\Delta = 0$; load condition (*a*)]
min $R_{8x} = -6.97 \cdot 10^4$ kg	[$\Delta \neq 0$; load condition (*a*), (*p*)]
max $R_{8x} = 6.50 \cdot 10^3$ kg	[$\Delta = 0$; load condition (*d*)]
max $R_{8x} = 7.60 \cdot 10^4$ kg	[$\Delta \neq 0$; load condition (*d*), (*q*)].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1/2 P
(a)
$\Delta = 0$
(b)
(c)
(d)
(l)
$\Delta \neq 0$
(m)
(n)
(o)
(p)
(q)

Fig. 8.

The local test then consists in comparing these results with the values compatible with the flexibility requirements for the structure, the mechanical properties of material and the characteristics of contraints.

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