Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

Santi Valenti, Maria Letizia Michelucci

## A condition on null sets for the approximate differentiability

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **56** (1974), n.5, p. 698–702.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_5_698_0>

**Topologia.** — *A condition on null sets for the approximate differentiability* (*). Nota di SANTI VALENTI (**) e MARIA LETIZIA MICHELUCCI (***), presentata (****) dal Socio G. SANSONE.

RIASSUNTO. — Si fornisce un'estensione del concetto di quasi assoluta continuità secondo Khintchine [1] e si mostra come una siffatta estensione consenta di assegnare, anche su di un insieme di L–misura nulla, una condizione caratteristica per la derivabilità approssimata quasi ovunque di una funzione reale di variabile reale.

## INTRODUCTION

Within the Notes [4] and [5], one of us (S.V.) has introduced an atypical set function (named " weight " or " relative weight "), endowed with positive values even on sets of L-measure 0. By means of this, a necessary and sufficient condition of (ordinary) differentiability has been given there for real functions. This condition exhibits the peculiarity of acting only on subsets of L-measure 0.

The aim of the present paper is to show that, by the aid of the same tools, an analogous condition can be found also for the approximate differentiability of real-valued functions defined on an interval $[a, b]$ of the real $\mathbf{R}$.

1. Let $f(x)$ be a function from $[a, b]$ to $\mathbf{R}$, with $[a, b] \subset \mathbf{R}$, and let X be a (proper or non-proper) subset of $[a, b]$. In the following, we shall put

$$\mathrm{O}_f(\mathrm{X}) = \underset{x \in \mathrm{X}}{\mathrm{l.u.b.}} [f(x)] - \underset{x \in \mathrm{X}}{\mathrm{g.l.b.}} [f(x)] ; \tag{1.1}$$

furthermore, we shall denote by $[\{\mathrm{T}_i\}]$ the class of all systems of intervals $\{\mathrm{T}_i\}$, $\mathrm{T}_i \subset [a, b]\ \forall i$, such that $i = 1, 2, \cdots, n$ ($n$ being an arbitrary natural number) and that

$$i \neq j \Rightarrow \mathrm{T}_i \cap \mathrm{T}_j = \varnothing . \tag{1.2}$$

Now, we shall put the following

DEFINITION 1.1. The function $f(x)$ is said to be *absolutely continuous* (K) (abbreviated ACK), on $\mathrm{E} \subset [a, b]$ iff for any $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that, whenever

$$\{\mathrm{T}_i\} \in [\{\mathrm{T}_i\}] \qquad \text{and} \qquad \sum_{i=1}^{n} |\mathrm{T}_i| < \delta , \tag{1.3}$$

(*) Address of the Authors: Institute of Mathematical Analysis, University of Palermo, Palermo (Italy).

(**) Partially supported by G.N.A.F.A. (Italian Research Council).

(***) Supported by a scholarship of M.P.I. (Ministero della Pubblica Istruzione).

(****) Nella seduta del 28 maggio 1974.

then

$$\sum_{i=1}^{n} O_f(E \cap T_i) < \varepsilon \quad (1.4)$$ (1).

By means of this, we can give the other

DEFINITION 1.2. The function $f(x)$ is said to be *almost absolutely continuous* (K) on $[a, b]$ iff for any $\varepsilon > 0$ there is in $[a, b]$ a corresponding perfect set P whose measure $|P|$ is greater than $b - a - \varepsilon$ and on which $f(x)$ is ACK. (We abbreviate this, saying that $f(x)$ is AACK).

On this base, one gives in [1] the following

THEOREM 1.1. *A necessary and sufficient condition in order that a measurable function $f(x)$ be almost everywhere approximately differentiable* (2) *on an interval $[a, b]$, is that $f(x)$ be* AACK *on this interval.*

2. Now, suppose that $f_0(x)$ be a real function defined on a subset Z of $[a, b]$, being Z dense on such an interval. It would be desirable to extend the last condition of approximate differentiability to the continuous extension (if it exists) $f(x)$ of $f_0(x)$ to the whole interval, even in the case that Z be of L-measure 0. Nevertheless, what would "almost" mean if the condition has to involve only Z? Furthermore, will an analogous statement hold in such a connection (namely within a possible, larger acception of the word "almost")?

As we are going to see, there is a complete answer to both these questions in terms of a special set function already introduced in [4] and [5], which is somewhat similar to the "outer measure" of sets according G. Peano and C. Jordan [3]. Therefore, we first report the following

DEFINITION 2.1. Let Z be a set contained in $[a, b]$ and let $Z' \subset Z$. Denote by $\{I\}_n$ any system of intervals obtained when $[a, b]$ is subdivided dichotomically into $2^n$ sub-intervals, $n$ being an arbitrary natural number. Then let us call $\{J\}_n (\{J'\}_n)$ the subsystem of $\{I\}_n$ consisting of all the intervals either containing points of Z (of Z′) in their interior, or having such a point as right endpoint. Finally, let N (N′) be the number of intervals belonging to $\{J\}_n$ (to $\{J'\}_n$). Iff the limit of the quantity N′/N exists as $n \to \infty$, then we shall say that the subset Z′ *has weight* relative to Z and we shall denote

(1) This definition and the following are essentially a translation of the analogous definitions given in [1] by A. Khintchine. However, we have to emphasize that the original idea of absolutely continuous function is due, at least for intervals, to G. Vitali, who first introduced this concept within his classic memoir quoted in reference [2].

(2) We recall that a function $f(x)$ is, by definition, *approximately differentiable* at $x_0 \in [a, b]$ when there is a number $f^*(x_0)$ such that, no matter what is $\varepsilon > 0$, the set

$$\left\{ x \in [a, b] : \left| \frac{f(x) - f(x_0)}{x - x_0} - f^*(x_0) \right| < \varepsilon \right\},$$

has $x_0$ among its *density points* [6].

this limit, namely the weight of Z′, by $w(Z')$. (There is no difficulty in proving that if Z is dense on $[a, b]$, then every subset Z′ has weight relative to Z and is found to be independent of the law of decomposition of the given interval, since one has

$$w(Z') = \frac{|\overline{Z}'|}{b-a}, \tag{2.1}$$

$\overline{Z}'$ being the closure of Z′).

Thereby, we can give the following

DEFINITION 2.2. Given a function $f_0(x)$ defined on a set $Z \subset [a, b]$ and such that $\overline{Z} = [a, b]$, this function will be said O-*absolutely continuous* (abbreviated OAC) on Z iff for any $\varepsilon > 0$ there is a corresponding subset Z′ of Z whose weight $w(Z')$ is greater than $1 - \varepsilon$ and on which $f_0(x)$ is ACK. (It is almost obvious that any function which is AACK on $[a, b]$ is OAC on this interval; *but the converse does not hold*).

At present, we are able to give answer to the questions proposed above, by means of the following two theorems. (It is worth noting that in the first of them the function $f_0(x)$ is assumed uniformly continuous on Z; this assumption is essential if we want to be sure that a unique continuous extension of $f_0(x)$ exists on $[a, b]$).

THEOREM 2.1. *Let* $Z \subset [a, b]$, $\overline{Z} = [a, b]$ *and let* $f_0(x)$ *be a function from* Z *to* **R**, *uniformly continuous on* Z. *If* $f_0(x)$ *is* OAC *on* Z, *then the continuous extension* $f(x)$ *of* $f_0(x)$ *to* $[a, b]$ *is almost everywhere approximately differentiable on this interval.*

*Proof.* Let $\varepsilon$ be an arbitrary positive number and let Z′ be any subset of Z such that

$$w(Z') > 1 - \varepsilon, \tag{2.2}$$

on which $f_0(x)$ be ACK. We must show that there is in $[a, b]$ a perfect set P such that $f(x)$ be ACK on it and, in addition, that

$$|P| > b - a - \eta, \tag{2.3}$$

$\eta$ being as arbitrary as $\varepsilon$ is.

Therefore, let us observe that

$$|\overline{Z}'| > (b-a)(1-\varepsilon); \tag{2.4}$$

besides, as is known, there are a perfect set P and a countable set N for which

$$\begin{cases} P \cap N = \varnothing, \\ P \cup N = \overline{Z}', \end{cases} \tag{2.5}$$

so that

$$|P| > (b-a)(1-\varepsilon). \tag{2.6}$$

On the other hand, by the continuity hypotheses, we have for any interval $T \subset [a, b]$

$$O_f(\bar{Z}' \cap T) = O_f(Z' \cap T) = O_{f_0}(Z' \cap T)\,, \tag{2.7}$$

whence, since obviously

$$O_f(P \cap T) \leq O_f(\bar{Z}' \cap T)\,, \tag{2.8}$$

we draw out that $f(x)$ is ACK on P. But putting

$$\eta = \varepsilon(b - a)\,, \tag{2.9}$$

by (2.6) we conclude that $f(x)$ is also AACK on $[a, b]$ and so approximately differentiable on this interval, a.e..

Conversely we have

THEOREM 2.2. *Let $f(x)$ be a measurable function from $[a, b]$ to $\mathbf{R}$; if such a function is almost everywhere approximately differentiable on this interval, then there is in $[a, b]$ a set* Z *of* L-measure 0 *and dense on $[a, b]$ such that the restriction on it of $f(x)$, say $f_0(x)$, there is* OAC.

*Proof.* As $f(x)$ is AACK on $[a, b]$, we can proceed as follows: set a number $\lambda_0 > 0$ and consider the sequence

$$\{\varepsilon_n\} = \left\{\frac{\lambda_0}{n}\,,\ n = 1\,, 2\,, \cdots\right\}; \tag{2.10}$$

then, for any $n$, denote by $P_n$ a perfect set submitted to these conditions

$$\begin{cases} P_n \subset [a, b]\,, \\ |P_n| > b - a - \varepsilon_n\,, \\ f(x) \text{ be ACK on } P_n\,. \end{cases} \tag{2.11}$$

After, still for any $n$, call $Z_n$ a set such that

$$\begin{cases} \bar{Z}_n = P_n\,, \\ |Z_n| = 0\,. \end{cases} \tag{2.12}$$

Clearly, it will be

$$(b - a)\, w(Z_n) = |P_n| > b - a - \varepsilon_n\,. \tag{2.13}$$

Finally, put

$$Z = \bigcup_{i=1}^{\infty} Z_i\,. \tag{2.14}$$

It can easily be seen that Z is dense on $[a, b]$; further, its measure is 0. Thus, we have to show that the restriction $f_0(x)$ of $f(x)$ to Z there is OAC. Then, set $\sigma > 0$ arbitrarily and choose the least integer, say $n_0$, for which the inequality

$$n\sigma > \lambda_0 \tag{2.15}$$

is satisfied; on the corresponding perfect set $P_{n_0}$, whose measure is greater than $b - a - \sigma$, the function $f(x)$ will be ACK. In addition, since for all intervals $T \subset [a, b]$ one has

$$O_f(P_{n_0} \cap T) \geq O_{f_0}(Z_{n_0} \cap T), \tag{2.16}$$

$f_0(x)$ will be ACK on $Z_{n_0}$. But from (2.13) we draw

$$w(Z_{n_0}) = \frac{|P_{n_0}|}{b-a} > 1 - \frac{\sigma}{b-a}. \tag{2.17}$$

So that, being $\sigma$ arbitrary, $f_0(x)$ is OAC.

*Remark.* Observe that in Theorem 2.1 we have not emphasized the possibility that Z have L-measure 0; while this circumstance has been underlined in Theorem 2.2. That is why, within the "sufficient" condition, we wished to point out that our statement contains, as a particular case, Theorem 1.1. On the contrary, within the "necessary" condition, all the strength of the statement consists just in the existence of a subset Z, *whose L-measure is* 0, on which the function is OAC.

## References


[1] KHINTCHINE A. (1927) – *Recherches sur la structure des fonctions mesurables*, « Fund. Math. », *9*, 212.

[2] VITALI G. (1905) – *Sulle funzioni integrali*, « Atti R. Acc. Sc. Torino », *40*, 1021.

[3] SANSONE G. (1954) – *Lezioni sulla teoria delle funzioni di variabili reali*, « Soc. Ed. Univ. », Firenze, 2nd printing, *35*.

[4] VALENTI S. (1970) – *On differentiability of real functions*, « Rend. Circ. Mat. Palermo », 2nd ser., *19*, 5.

[5] VALENTI S. (1973) – *On the differentiability of functions whose values are assigned on* 0-*measure sets*, « Rev. Roum. Math. Pures et Appliquées », *18*, 1283.

[6] HARTMAN S. and MIKUSIŃSKI J. (1961) – *The theory of Lebesgue measure and integration*, Pergamon Press and Państwowe Wydawnictwo Naukowe, *98*.