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An approximation theorem on CR-submanifold of a complex manifold

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Geometria differenziale. — *An approximation theorem on CR-submanifold of a complex manifold* (*). Nota di GILBERTO DINI e MARIO LANDUCCI, presentata (**) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si enuncia un teorema di approssimazione sulle CR–sottovarietà di una varietà complessa. La dimostrazione comparirà in altro lavoro.

Let X be a complex manifold of complex dimension n, U an open subset of X, we denote with $A_{\bar\partial}^{p,q}(U)$ the sheaf of germs of $\bar\partial$-closed differentiable forms of type (p, q) on U.

We say that a real differentiable submanifold M of X is a CR-submanifold with CR dimension m if $\dim_{\mathbf{C}} H_x(M) = m$ for every $x \in M$, where $H_x(M)$ is the maximal complex subspace of the tangent space to M at x, carrying the same complex structure of $T_x(X)$.

In this Note we sketch the proof of the following theorem:

THEOREM 1. *If* M *is a compact* CR *submanifold of* X *with* CR *dimension* m, *and if* $C^{p,m}(M)$ *is the space of continuous sections of the sheaf* $A^{p,m}(M)$, *then there exists a neighborhood of* M *in* X *s.t. the inclusion map:*

$$A_{\bar\partial}^{p,m}(U) \to C^{p,m}(M)$$

has dense range in the C^∞ *topology.*

The result generalizes a theorem proved by Nirenberg-Wells [3]. The proof is divided into four steps.

Step A: Let h_{ij} be an hermitian metric on X, locally euclidean; as in [3] we prove that there exists a neighborhood U of M in X such that for every $f \in A^{p,m}(M)$ and for every $N \in \mathbf{N}$ we can find $\tilde f \in A^{p,m}(U)$ s.t.

i) $\tilde f|_M = f$;

ii) $|\bar\partial \tilde f| = o(d_M^N)$

where d_M is the distance from M.

Step B: We recall that a complex manifold X is q-complete if there exists a C^2 function φ s.t. its Levi form has $n - q + 1$ strictly positive eigenvalues at every $x \in X$ and the sets $B_c = \{x : \varphi < c\}$ are relatively compact subsets of X for every $c \in \mathbf{R}$.

It can be proved that for some l, $T(l) = \{z \in X : d_M^2(z) < l^2\}$ is a relatively compact m-complete neighborhood of M.

(*) Lavoro rientrante nell'ambito dell'attività del C.N.S.A.G.A. del CNR.
(**) Nella seduta del 28 maggio 1974.

In what follows we will denote with $\| \ \|_{U,k}$ the usual Sobolev norms and, as usual, if K is a compact set and s a nonnegative integer

$$| \ |_{K,s} = \sup_{K,|\alpha|\leq s} |D^\alpha| .$$

Step C: THEOREM 2 (Hormander). *For every* $f \in A_{\bar\partial}^{p,m+1}(T(l))$ *there exists* $u_l \in A^{p,m}(T(l))$ *s.t.*

i) $\bar\partial u_l = f$;

ii) $\| u_l \|_{T(l),0} \leq C \| f \|_{T(l),0}$

where C, for sufficiently small l, does not depend on l.

The proof follows by a careful analysis of the paper [2].

Step D: Outline of the proof of Theorem 1.

By a theorem of Sorani [4], asserting that if B_c and B_d, $c > d$ are as in step B then the restriction map

$$A_{\bar\partial}^{p,m}(B_c) \to A_{\bar\partial}^{p,m}(B_d)$$

is dense in the topology of uniform convergence on compact subsets, it is sufficient to prove that the map

$$A_{\bar\partial}^{p,m}(M) \to A^{p,m}(M)$$

is dense.

Let $f \in A^{p,m}(M)$, for every $\varepsilon > 0$, $k \geq 0$ must find $h \in A_{\bar\partial}^{p,m}(M)$ s.t.

$$| h - f |_{M,k} < \varepsilon .$$

We choose, by step A, the extension $\tilde f$ s.t. $\bar\partial \tilde f = o(d_M^N)$ with $N > 7n + 3k + 2$.

By steps B and C we can find $u_l \in A^{p,m}(T(l))$ s.t.

i) $\bar\partial u_l = \bar\partial \tilde f$;

ii) $\| u_l \|_{T(l),0} \leq C \| \bar\partial \tilde f \|_{T(l),0}$.

Using some well-known estimates (see [3], Prop. 5.1, 5.2, 5.3, 5.4) we have

$$| u_l |_{M,k} \leq C\, l^{-(3n+k+1)} \| u_l \|_{T(l/2),n+k+1} \leq C' l^{-(7n+3k+2)} \| \bar\partial \tilde f \|_{T(l),0} .$$

So $h = f - u_l$ satisfies the thesis, for sufficiently small l.

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