Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

James O. C. Ezeilo

Some new criteria for the existence of periodic solutions of a certain second order differential equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.5, p. 675–683.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_5_675_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Equazioni differenziali ordinarie. — *Some new criteria for the existence of periodic solutions of a certain second order differential equation.* Nota di JAMES O. C. EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si considera l'equazione (1) $\ddot{x} + f(x)\dot{x} + g(x) = q(t)$, dove $f(x)$, $g(x)$, $q(t)$ sono funzioni continue dei loro argomenti, $f(x) \geq a > 0$, $q(t + \omega) = q(t)$ per tutti i t e $\int_0^\omega q(s)\,ds = 0$ per qualche ω.

Sia $H = \left\{ \varphi \in C^1[0, \omega] : \varphi(t + \omega) = \varphi(t) \text{ qualunque sia } t \text{ e } \int_0^\omega \varphi(s)\,ds = 0 \right\}$. Si dimostra che se g è tale che $\int_0^\omega g(\varphi(s))\,ds = 0$ per ogni $\varphi \in H$, esiste allora almeno una soluzione ψ di (1) di periodo ω e $\int_0^\omega \psi(s)\,ds = 0$.

1. INTRODUCTION

We shall examine here the existence of periodic solutions of the second order differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = q(t) \tag{1.1}$$

in which f, g, q are continuous functions depending only on the arguments shown and $g(t)$ is ω-periodic in t, that is $q(t + \omega) = q(t)$, for all t.

The problem has already been very exhaustively investigated in the two main cases:

(I) When (1.1) is parameter dependent and f, g are sufficiently smooth, using the well known analytical techniques of Poincare, Krylov, Bogoliubov and others.

(II) When (1.1) is of the dissipative type, that is, having all its solutions ultimately bounded by a constant whose magnitude depends only on f, g and q. The second order equation

$$\ddot{x} + a\dot{x} + bx = q(t), \tag{1.2}$$

in which a and b are constants, is in this latter category (II) if

$$a > 0 \quad \text{and} \quad b > 0, \tag{1.3}$$

and the conditions on f and g which have dominated the investigation of (1.1) under the category (II) are generalizations in some form, or other, of (1.3).

(*) Nella seduta del 28 maggio 1974.

The interested reader is referred to [1], [2], [3] and [4] for an extensive review of the techniques, results and other generalizations of the existence problem when (1.1) is of the category (I) or (II), as well as for other relevant references.

The present treatment of (1.1) has a somewhat different motivation from the categories (I) and (II), and concerns the special case in which $\int_0^t q(s)\,ds$ is bounded for all t, or, what is the same thing,

$$\int_0^\omega q(t)\,dt = 0 . \tag{1.4}$$

Assume indeed that (1.4) holds and let $\varphi(t)$ be any ω-periodic solution of (1.1). Then clearly

$$\int_0^\omega g(\varphi(t))\,dt = 0 . \tag{1.5}$$

It is however easy to see that (1.5) by itself does not in general imply that $\varphi(t)$, even if ω-periodic, is a solution of (1.1), and it is thus of interest to investigate whether, and under what further conditions of f and g, (1.5) can be used as a basis for the proof of the existence of ω-periodic solutions of (1.1). One result in this direction which I have been able to establish, and which will be proved here, is the following:

THEOREM. *Assume that* (1.4) *holds and that* $f(x) \geq a$ *(all* x*) for some constant* $a > 0$. *Let*

$$H = \left\{ \varphi \in C^1[0, \omega] : \ \varphi \textit{ is } \omega\textit{-periodic in } t \textit{ and } \int_0^\omega \varphi(t)\,dt = 0 \right\} .$$

If g *is such that* (1.5) *holds for each* $\varphi \in H$, *then there exists at least one* ω-*periodic solution* φ^* *of* (1.1) *which also satisfies* $\int_0^\omega \varphi^*(t)\,dt = 0$.

2. REMARKS ON THEOREM

The functions $g \equiv 0$, $g = bx$ (b constant) are the more obvious examples of functions g which satisfy (1.5) for each $\varphi \in H$. By considering the Fourier expansion

$$\varphi \sim \sum_{n=1}^{\infty} \left(\alpha_n \sin \frac{2\pi nt}{\omega} + \beta_n \cos \frac{2\pi nt}{\omega} \right) \tag{2.1}$$

for each $\varphi \in H$ $\left(\text{note here, by the way, that } \beta_0 = 0, \text{ since } \int_0^\omega \varphi\,dt = 0\right)$ it can however be verified that each of the following functions also meets the given requirements on g:

(i) $\sum_{k-1}^{n} c_k x^{2k+1}$ (n finite integer; $c_1, \cdots c_n$, all constants);

(ii) $x^{2m} \sin x$ ($m \geq 0$ integer);

(iii) $f_n(x)$ defined inductively, for any integer $n \geq 0$, by $f_0(x) = \sin x$, $f_{n+1}(x) = \sin(f_n(x))$.

In the sense that, when $g = bx$ (with b constant) (1.5) is trivially fulfilled for all $\varphi \in \mathrm{H}$, regardless of the sign of b, the theorem gives a better generalization of the existence result for the constant-coefficient equation (1.2) than any of the theorems for equations (1.1) of the dissipative type which cover only the case $b > 0$ in (1.2). Notwithstanding this however, it will not be correct to say that the present theorem generalizes any of the standard existence theorems for equations (1.1) of the dissipative type, and vice versa, as there are functions $g(x)$ which satisfy the dissipativity condition $g(x)/x > 0$ for all sufficiently large $|x|$, but which do not satisfy the condition on g in the present theorem, and vice versa. For example the function g_0 defined by

$$g_0(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ x, & \text{if } x \leq 0 \end{cases}$$

clearly satisfies the dissipativity condition $g_0(x)/x > 0$ $(x \neq 0)$ and yet

$$\int_0^{\omega} g_0(\sin(2\pi\omega^{-1} t))\,\mathrm{d}t = \pi^{-1}\omega .$$

On the other hand $g_1(x) = \sin x$, which satisfies the condition of the present theorem, does not satisfy the dissipativity condition $g_1(x)/x > 0$ for all sufficiently large $|x|$.

3. PRELIMINARIES OF PROOF

The procedure is by the Leray-Schauder fixed point technique, although we shall specifically use Schaefer's version of the fixed point theorem (given in [5]).

A convenient starting point is the parameter (μ)-dependent equation

$$\ddot{x} + \{(1-\mu)a + \mu f(x)\}\dot{x} + (1-\mu)\delta x + \mu g(x) = \mu q(t) \quad (0 \leq \mu \leq 1) \tag{3.1}$$

which reduces to a constant coefficient when $\mu = 0$ and to the original equation (1.1) when $\mu = 1$. Here δ is an arbitrarily chosen, though fixed, positive constant. Note that the equation (3.1) can be represented more compactly in the 2-vector form

$$\dot{\mathrm{X}} = \mathrm{AX} + \mu\mathrm{G}(\mathrm{X}, t) \tag{3.2}$$

where the vectors X , G and the matrix A are given by

$$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\delta & -a \end{bmatrix}, \quad G = \begin{pmatrix} 0 \\ -(f(x) - a)\,\dot{x} - (g(x) - \delta x) + q(t) \end{pmatrix}.$$

We introduce now a vector function

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

defined, for any $\varphi \in H$, by

$$\Psi(t) = \int_0^{\omega} e^{(t-s)A}\, G(\varphi(s), s)\, ds . \tag{3.3}$$

This infinite integral is quite standard in the investigation of periodic solutions of (3.2) in cases where the matrix A has all its eigenvalues with negative real parts, as is indeed the case here since a and δ are both positive. The reader is particularly referred to the arguments in § 4.2 of [6] which can be carried over, with very slight changes, to show that Ψ is well defined and has the following periodicity and smoothness properties:

(P_1) $\Psi(t)$ *is* ω-*periodic in* t *for each* $\varphi \in H$;

(P_2) $\frac{d}{dt}\Psi(t)$ *exists and satisfies*

$$\frac{d}{dt}\Psi(t) = A\Psi(t) + G(\varphi(t), t) . \tag{3.4}$$

Consider now the mapping T on H defined by

$$(T\varphi)(t) = \psi_1(t) \qquad (\varphi \in H) .$$

It is not difficult to see that $T : H \to H$. Indeed by (P_1), ψ_1 is ω-periodic and by (P_2) $\dot{\psi}_1$ exists and is continuous. Finally, by the definitions of A and G, we have from (3.4) that

$$\dot{\psi}_1 = \psi_2 \tag{3.5}$$

$$\dot{\psi}_2 = -\delta\psi_1 - a\psi_2 - \{f(\varphi) - a\}\,\dot{\varphi} - \{g(\varphi) - \delta\varphi\} + q(t) . \tag{3.6}$$

Integrating (3.5) from 0 to ω and using the ω-periodicity of ψ_1 gives that $\int_0^{\omega} \psi_2\, dt = 0$. Similarly, from (3.6), we have that

$$0 = -\delta \int_0^{\omega} \psi_1\, dt - a \int_0^{\omega} \psi_2\, dt = 0,$$

since $\int_0^{\omega} q(t)\, dt = 0$ and (1.5) holds for each $\varphi \in H$. Thus we have that $\int_0^{\omega} \psi_1\, dt = -a\delta^{-1} \int_0^{\omega} \psi_2\, dt = 0$, so that $T : H \to H$ as was asserted. But the main feature of the mapping T which is of special relevance to the existence problem in hand is the result that

LEMMA 1. *Any* $\varphi \in H$ *satisfying the functional equation*

$$\varphi = \mu T \varphi \tag{3.7}$$

is necessarily a solution of (3.1).

Proof of lemma. Indeed if $\varphi \in H$ satisfies (3.7) then $\dot{\varphi} = \mu \frac{d}{dt} T\varphi$ so that, by (3.5) and (3.6)

$$\ddot{\varphi} = -\delta\varphi - a\dot{\varphi} - \mu\{f(\varphi) - a\}\dot{\varphi} - \mu\{g(\varphi) - \delta\varphi\} + \mu q(t)$$

which shows that φ is a solution of (3.1).

It will be shown in the next three sections that, subject to the given conditions on f, g and q, there exists indeed at least one $\varphi \in H$ satisfying

$$\varphi = T\varphi \tag{3.8}$$

and the existence theorem for (1.1) will then follow from Lemma 1, since (3.1) reduces to (1.1) when $\mu = 1$.

4. For further progress in the proof of (3.8), it is convenient to consider H now as a normed linear space with the operations of addition and multiplication by a scalar defined in the usual way, and with the norm $\|\cdot\|$ defined by

$$\|\varphi\| = \sup_{0 \leq t \leq \omega} \{|\varphi(t)| + |\dot{\varphi}(t)|\}.$$

It is a simple matter, by proceeding in almost the same way as in § 4.4 of [6], to verify that the mapping $T : H \to H$ here is completely continuous. Thus the existence of a $\varphi \in H$ satisfying (3.8) will follow, as in [6], from Schaefer's theorem (cited earlier in § 3) if it can be shown that there exists a constant $D > 0$, independent of μ, such that

$$\|\varphi\| \leq D \tag{4.1}$$

for every $\varphi \in H$ satisfying (3.7) $(0 \leq \mu \leq 1)$.

Actually since (see Lemma 1) every $\varphi \in H$ which satisfies (3.7) necessarily satisfied (3.1), what we shall do here is to establish that every solution $\varphi \in H$ of (3.1) satisfies

$$|\varphi(t)| + |\dot{\varphi}(t)| \leq D \qquad \tau \leq t \leq \tau + \omega \tag{4.2}$$

for some τ, and the theorem will then follow since (4.2) implies (4.1).

5. *Boundedness of* $|\varphi(t)|$. Set

$$Q_0 = \max |q(t)| \; (t \in [0, \omega]).$$

In what follows hereafter the capitals D (with or without a suffix) will denote positive constants whose magnitude depends only on $\omega, a, \delta, Q_0, f$

and g, *but not on* μ. The D's are not necessarily the same each time they occur unless suffixes are attached, but the numbered D's: $D_0, D_1, D_2, \cdots$ retain the same identity throughout.

We shall require the following subsidiary results:

LEMMA 2. *For each* $\varphi \in H$

$$\int_0^\omega \varphi^2(t)\,dt \leq D_0 \int_0^\omega \dot{\varphi}^2(t)\,dt. \tag{5.1}$$

Proof. If φ has the Fourier Series expansion (2.1) then

$$\dot{\varphi} \sim 2\pi\omega^{-1} \sum_{n=1}^{\infty} \{n\alpha_n \cos(2\pi n\omega^{-1} t) - n\beta_n \sin(2\pi n\omega^{-1} t)\}. \tag{5.2}$$

By Parseval's Theorem applied to (2.1),

$$\int_0^\omega \varphi^2(t)\,dt = \frac{1}{2}\,\omega \sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2).$$

Similarly from (5.2) we have that

$$\begin{aligned}\int_0^\omega \dot{\varphi}^2(t)\,dt &= 2\pi^2\omega^{-1} \sum_{n=1}^{\infty} n^2(\alpha_n^2 + \beta_n^2)\\ &\geq 2\pi^2\omega^{-1} \sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2)\\ &= 4\pi^2\omega^{-2} \int_0^\omega \varphi^2(t)\,dt\end{aligned}$$

which proves (5.1) with $D_0 = (1/4)\,\omega^2\pi^{-2}$.

LEMMA 3. *There exists* D_1 *such that, for any fixed* τ *and any* $\varphi \in H$ *satisfying* (3.1)

$$\int_\tau^{\tau+\omega} \dot{\varphi}^2(t)\,dt \leq D_1. \tag{5.3}$$

Proof. Given $\varphi \in H$, define V by

$$2V = \dot{\varphi}^2 + (1-\mu)\,\delta\varphi^2 + \mu \int_0^\varphi g(s)\,ds.$$

If φ is a solution of (3.1), an elementary calculation will show that

$$\dot{V} = -\{(1-\mu)\,a + \mu f(\varphi)\}\,\dot{\varphi}^2 + \mu q(t)\,\dot{\varphi}$$

so that, since $f \geq a$ and $0 \leq \mu \leq 1$ imply that

$$(1 - \mu)\, a + \mu f(\varphi) \geq a\,,$$

we have

$$\begin{aligned}\dot{V} &\leq - a\dot{\varphi}^2 + Q_0\,|\,\dot{\varphi}| \\ &\leq - \frac{1}{2}\, a\dot{\varphi}^2 + D.\end{aligned}$$

Integrating both sides from τ to $\tau + \omega$, and noting that V is ω-periodic in t, since φ is ω-periodic in t, we obtain that

$$0 \leq - \frac{1}{2}\, a \int_{\tau}^{\tau+\omega} \dot{\varphi}^2\, at + D\omega$$

from which (5.3) now follows.

The boundedness of $|\,\varphi(t)\,|$, for each $\varphi \in H$ satisfying (3.1) can now be established. First of all, note that if $\varphi \in H$ satisfies (3.1) then

$$|\,\varphi(\tau_0)\,| \leq (D_0\, D_1\, \omega^{-1})^{1/2} \tag{5.4}$$

for some τ_0. For otherwise, that is if $|\,\varphi(t)\,| > D_0\, D_1\, \omega^{-1}$ for all t, then

$$\int_0^{\omega} \varphi^2(t)\, dt > D_0\, D_1$$

contradicting the estimate

$$\int_0^{\omega} \varphi^2(t)\, dt \leq D_0\, D_1$$

obtainable from combining (5.1) and (5.3).

Next, from the identity

$$\varphi(t) = \varphi(\tau_0) + \int_{\tau_0}^{t} \dot{\varphi}(s)\, ds$$

we have that

$$\begin{aligned}|\,\varphi(t)\,| &\leq |\,\varphi(\tau_0)\,| + \int_{\tau_0}^{t} |\,\dot{\varphi}(s)\,|\, ds \\ &\leq |\,\varphi(\tau_0)\,| + \omega^{1/2}\Big(\int_{\tau_0}^{\tau_0+\omega} \dot{\varphi}^2(s)\, ds\Big)^{1/2} \quad (\tau_0 \leq t \leq \tau_0 + \omega)\end{aligned}$$

by Schwarz's inequality; so that by (5.3) and (5.4)

$$|\,\varphi(t)\,| \leq (D_0\, D_1\, \omega^{-1})^{1/2} + (D_1\, \omega)^{1/2} = D_2 \qquad (\tau_0 \leq t \leq \tau_0 + \omega).$$

Since φ is ω-periodic, it follows that

$$|\,\varphi(t)\,| \leq D_2 \qquad \text{for all } t\,. \tag{5.5}$$

6. *Boundedness of* $|\dot{\varphi}(t)|$. Let $\varphi \in \mathrm{H}$ be any solution of (3.1). We shall prove here the existence of some $\mathrm{T} \geq 0$ such that

$$|\dot{\varphi}(t)| \leq \mathrm{D}, t \geq \mathrm{T}$$

and the required boundedness of $\dot{\varphi}$ for all t will then follow since $\dot{\varphi}$ is ω-periodic.

Define $\mathrm{W} = \mathrm{W}(t)$ by

$$2\,\mathrm{W} = \dot{\varphi}^2(t).$$

Since φ satisfies (3.1) we have that

$$\dot{\mathrm{W}} = -\{(1-\mu)\,a + \mu f(\varphi)\}\,\dot{\varphi}^2 - \{(1-\mu)\,\delta\varphi + \mu g(\varphi) - \mu q\}\,\dot{\varphi}.$$

As before,

$$(1-\mu)\,a + \mu f(\varphi) \geq a > 0.$$

Also, by (5.5) and since $0 \leq \mu \leq 1$ and q is bounded,

$$|(1-\mu)\,\delta\varphi + \mu g(\varphi) - \mu q| \leq \mathrm{D}.$$

Hence

$$\begin{aligned} \dot{\mathrm{W}} &\leq -a\dot{\varphi}^2 + \mathrm{D}\,|\dot{\varphi}| \\ &\leq -1, \qquad \text{if } |\dot{\varphi}| \geq \mathrm{D}_3 \end{aligned} \tag{6.1}$$

for some sufficiently large D_3. It is easy to see from this that

$$|\dot{\varphi}(\tau_1)| < \mathrm{D}_3 \tag{6.2}$$

for some τ_1. Indeed if otherwise we had that $|\dot{\varphi}(t)| \geq \mathrm{D}_3$ for all t, then by (6.1) $\dot{\mathrm{W}} \to -\infty$ as $t \to \infty$, which is impossible, W being non negative.

We assert now that, with τ_1 determined by (6.2),

$$|\dot{\varphi}(t)| \leq 2\,\mathrm{D}_3 \qquad \text{for all} \quad t \geq \tau_1. \tag{6.3}$$

For suppose that this were not so. Then, by (6.2) and (6.3), $\dot{\varphi}(t)$ being continuous, there exist t_1, t_2 with $t_2 > t_1 > \tau_1$ such that

$$|\dot{\varphi}(t_1)| = \mathrm{D}_3 \quad , \quad |\dot{\varphi}(t_2)| = 2\,\mathrm{D}_3 \tag{6.4}$$

and

$$|\dot{\varphi}(t)| \geq \mathrm{D}_3 \qquad (t_1 \leq t \leq t_2). \tag{6.5}$$

But, by the definition of W, (6.4) implies that

$$\mathrm{W}(t_2) = 2\,\mathrm{D}_3^2 > \frac{1}{2}\,\mathrm{D}_3^2 = \mathrm{W}(t_1),$$

which clearly contradicts the inequality

$$W(t_2) < W(t_1)$$

implied by (6.1) if $\dot{\varphi}$ is subject to (6.5). This proves (6.3) and hence also the required boundedness of $|\dot{\varphi}(t)|$ for all t.

With the result (4.2) thus completely established for any $\varphi \in H$ satisfying (3.1), the theorem now follows as was pointed out in § 4.

REFERENCES

[1] L. CESARI (1963) – *Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations*, Springer Verlag, Berlin.
[2] S. LEFSCHETZ (1957) – *Differential Equations: Geometric Theory*, Interscience, New York.
[3] R. REISSIG, R. CONTI and G. SANSONE (1963) – *Qualitative Theorie Nichtlinearer Differential gleichungen*, Edizioni Cremonese, Rome.
[4] J. HALE (1963) – *Oscillations in Nonlinear Systems*, McGraw Hill, New York.
[5] J. CRONIN (1964) – *Fixed points and Topological Degree in Nonlinear Analysis*, American Mathematical Society, p. 133.
[6] J. O. C. EZEILO and H. O. TẸJUMỌLA (1966) – «Ann. Mat. Pura Appl.», *74*, 283–316.