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An Extremal Principle for Accounting Balance of a Resource Value-Transfer Economy: Existence, Uniqueness and Computation

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Ricerca operativa. — *An Extremal Principle for Accounting Balance of a Resource Value-Transfer Economy: Existence, Uniqueness and Computation.* Nota di ABRAHAM CHARNES e WILLIAM W. COOPER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota si sviluppa un principio estremale per un problema di contabilità economica di W. P. Drews. Le condizioni di esistenza e unicità vengono caratterizzate completamente mediante un problema di programmazione lineare di tipo distribuzione pura (trasporto).

Un teorema di esistenza di Dantzig (stabilito tramite il teorema del punto fisso di Brouwer sotto condizioni economiche incerte) appare come un corollario immediato.

Il principio è equivalente a una minimizzazione non vincolata di una semplice funzione strettamente convessa, tale quindi da rendere facili i calcoli. Il principio porta ad un problema di programmazione geometrica il cui duale è un problema di minimizzazione di una funzione strettamente convessa soggetta alla distribuzione di vincoli dianzi menzionata.

1. INTRODUCTION

Events of recent years in re-exhaustible resources led W. P. Drews to formulate a model (which we have not seen) in which an economy is driven by a transfer of agreed monetary values of r resources, e.g. oil, food, labor, etc., into agreed monetary receipts of s resource-owner (= goods-consumer) groups such that (1) total resource valuation equals total monies received, (2) prices of resources and sizes of owner groups adjust to drive consumer goods prices and industrial activity levels (employment levels) so that the value of each resource equals the sum the economy spends for it and the receipts of each owner-group equals the sum of its expenditures. The latter detail (2) we call " accounting balance ".

A variant of Drews' model was presented and studied by Dantzig in [1]. He shows only, by making a strong (and economically questionable) assumption on a matrix relating resource use to owner group size and employing the Brouwer fixed point theorem, that relative prices and relative group sizes exist which satisfy the accounting balance. No possibilities of computation are suggested, the question of uniqueness is untouched, and nothing is mentioned concerning the effect of the structure of the matrix or the agreed monetary divisions on the balance.

In the following we completely characterize such an economy by a non-linear extremal principle which may be interpreted as the minimization of an economic potential function. Accounting balance occurs if and only if

(*) Nella seduta del 20 aprile 1974.

there exists a point of economic equilibrium (minimum potential). Despite the non-linearity of the functional minimized, the effects of the key $(r \times s)$ matrix M, the resource valuations $\gamma_1, \cdots, \gamma_r$ and the owner receipts $\delta_1, \cdots, \delta_s$ on the nature of the minimum (non-existence, existence, uniqueness) are completely characterized in terms of an associated linear programming problem of distribution (transportation) type with supplies $\gamma_1, \cdots, \gamma_r$ receipts $\delta_1, \cdots, \delta_s$ and with the non-zero entries of M designating the possible " transfer " routes. The extended dual theorem, p. 182, [2], and dual considerations are the key here.

The extremal problem is reduced to the *unconstrained* minimization of a strictly convex function involving only exponential and linear terms. Computation is easily possible with SUMT, piecewise linearization, or any other standard method. Alternately it can be viewed as an extended geometric programming problem, whose dual is a convex programming problem with pure distribution constraints. The latter formulation seems most advantageous to compute large systems.

2. NOTATION AND FORMULATION

Let ${}_iA$, A_j be respectively the i^{th} row, j^{th} column of the matrix A. Let p^T, q be the row, column vector of prices per unit resource, sizes of owner (consumer) groups; y, x the vectors of goods prices, industry activity (employment) levels; R, the $(r \times s)$ matrix of resources per unit industrial activity, C the $(m \times s)$ matrix of goods per unit owner (consumer) group size, N the $(n \times m)$ matrix of industrial activity per unit good, in [1] the inverse of a Leontief type matrix); $M \equiv RNC$. We assume $\gamma_i, \delta_j > 0$ for all i, j. Note that R, N, C are non-negative matrices,

$$y^T = p^T RN \quad , \quad x = NCq, \tag{2.1}$$

$$\sum_i \gamma_i = \sum_j \delta_j \quad \text{i.e. the total value is transferred.} \tag{2.2}$$

The accounting balance conditions are

$$\gamma_i = p_i ({}_iMq) \quad , \quad \delta_j = (p^T M_j) q_j , \qquad \text{all } i, j. \tag{2.3}$$

LEMMA 1. *If* $\gamma_i, \delta_j > 0$, *for all* i, j *and* (2.3) *holds then* $p_i = e^{u_i}$, $q_j = e^{v_j} j > 0$, *for all* i, j, *and* M *has a positive entry in each row and each column.*

LEMMA 2. *At accounting balance,*

$$\sum_i \gamma_i = p^T M q = \sum_j \delta_j .$$

Then (2.3) *may be rewritten*

$$\sum_j M_{ij} e^{u_i + v_j} = \gamma_i \quad , \quad \sum_i M_{ij} e^{u_i + v_j} = \delta_j , \qquad \text{all } i, j. \tag{2.4}$$

Let

$$\mathscr{C}(u, v) \equiv \sum_{i,j} M_{ij} e^{u_i+v_j} - \sum_i \gamma_i u_i - \sum_j \delta_j v_j . \tag{2.5}$$

Then, by the Kuhn-Tucker Theorem, since $\mathscr{C}(u, v)$ is convex and analytic in (u, v),

THEOREM 1. *$\mathscr{C}(u, v)$ has a minimum at (u, v) if and only if the accounting balance* (2.4) *holds.*

The problem of balance may therefore be rendered by the *Extremal Principle*:

$$\text{minimize } p^{\mathrm{T}} \mathrm{M} q - \sum_i \gamma_i \ln p_i - \sum_j \delta_j \ln q_j , \tag{2.5}$$

with $p^{\mathrm{T}}, q > 0$, or equivalently

$$\text{minimize } \mathscr{C}(u, v) \equiv \sum_{i,j} M_{ij} e^{u_i+v_j} - \sum_i \gamma_i u_i - \sum_j \delta_j v_j \tag{2.6}$$

without constraints.

Note: (2.5) may be recognized as an extended geometric programming problem.

3. EXISTENCE

The necessary condition on M in Lemma 1 is *not* sufficient even that $\mathscr{C}(u, v)$ be bounded below. To characterize this situation, consider the following "transfer" system (of "pure distribution" type in linear programming jargon):

$$\begin{aligned} &\sum_{j \in \mathrm{J}(i)} x_{ij} = \gamma_i , && i = l, \cdots, r \\ &\sum_{i \in \mathrm{I}(j)} x_{ij} = \delta_j , && j = l, \cdots, s \\ &x_{ij} \geq 0, \end{aligned} \tag{3.1}$$

where $\mathrm{J}(i) \equiv \{j : M_{ij} > 0\}$, $\mathrm{I}(j) \equiv \{i : M_{ij} > 0\}$.

THEOREM 2. *$\mathscr{C}(u, v)$ is bounded below if and only if the transfer system* (3.1) *has a solution.*

Proof. If (3.1) has a solution $\bar{x}_{ij}$, we can rewrite (2.6) as

$$\mathscr{C}(u, v) = \sum_{i,j} M_{ij} e^{u_i+v_j} - \sum_{i,j} \bar{x}_{ij} (u_i + v_j) \tag{3.2}$$

and get

$$\mathscr{C}(u, v) \geq \sum_{(i,j) \in \mathrm{P}} \bar{x}_{ij} [1 - \ln (\bar{x}_{ij}/M_{ij})] , \tag{3.3}$$

where $\mathrm{P} \equiv \{(i, j) : \bar{x}_{ij} > 0\}$.

If (3.1) has no solution, we minimize $\sum_{i,j} 0 \cdot x_{ij}$ subject to the transfer conditions. The dual thorem [2] implies that a sequence (u^n, v^n) exists for which $\mathscr{C}(u^n, v^n) \to -\infty$, *q.e.d.*

Let D be the coefficient matrix of the left side of the equations in (3.1). Its maximum possible rank is $r + s - 1$. By the " regularization " procedure of Charnes and Cooper [2] all cases can be reduced (also computationally!) to this case. We discuss this elsewhere.

Here we note that the existence of a solution to (3.1), while sufficient for an infimum, is *not* sufficient for existence of a minimum of $\mathscr{C}(u, v)$.

THEOREM 3. *$\mathscr{C}(u, v)$ has a minimum (= accounting balance) if and only if the transfer system (3.1) has a solution $\bar{x}_{ij}$ whose non-zero values designate a sub-matrix of D of equal rank to D.*

Proof. Suppose (3.1) has a solution $\bar{x}_{ij}$ whose non-zero components designate a submatrix of rank less than D. Let $w_{ij} = u_i + v_j$ for $\bar{x}_{ij} > 0$. Let $z_{kl} = u_k + v_l$ if $d_{kl} > 0$ and $\bar{x}_{kl} = 0$. Then the z_{kl} are independent variables i.e. the $u_k + v_l$ cannot be expressed in terms of the w_{ij}. We can now write

$$(3.4) \qquad \mathscr{C}(u, v) = \tilde{\mathscr{C}}(w, z) \equiv \sum_{\bar{x}_{ij}>0} (M_{ij} e^{w_{ij}} - \bar{x}_{ij} w_{ij}) + \sum_{(k,l)\in K} M_{kl} e^{z_{kl}},$$

where $K \equiv \{(k, 1) : M_{kl} > 0, \ \bar{x}_{kl} = 0\}$.

Clearly, $\tilde{\mathscr{C}}(w, \bar{z}) < \tilde{\mathscr{C}}(w, z)$ whenever $\bar{z} \leq z$ and some $\bar{z}_{kl} < z_{kl}$. Thus $\mathscr{C}(u, v)$ has an infimum and no minimum whenever $K \neq \varnothing$. If $K = \varnothing$, then

$$(3.5) \qquad \mathscr{C}(u, v) = \sum_{\bar{x}_{ij}>0} (M_{ij} e^{w_{ij}} \bar{x}_{ij} w_{ij}) \geq \sum_{(i,j)\in\hat{Q}} (M_{ij} e^{w_{ij}} - \hat{x}_{ij} w_{ij}) \equiv \bar{\mathscr{C}}(w)$$

where $\hat{x}_{ij}$ satisfies (3.1), $\hat{Q} \equiv \{(i, j) : \hat{x}_{ij} > 0\} \subseteq \{(i, j) : \bar{x}_{ij} > 0\}$ and $\hat{Q}$ designates linearly independent columns of D, hence independent variables w_{ij}.

But $\bar{\mathscr{C}}(w) \to \infty$ as $|w| \to \infty$. Since $K = \varnothing$ if and only if the rank of the submatrix of D designated by $\bar{x}_{ij} > 0$ equals that of D, the theorem follows.

Dantzig's result is the

COROLLARY 1. *If $M > 0$, $\mathscr{C}(u, v)$ has a minimum.*

COROLLARY 2. *If $P \equiv \{(i, j) : \bar{x}_{ij} > 0\} = Q \equiv \{(i, j) : M_{ij} > 0\}$ and the columns of D designated by P are linearly independent, then the minumum of $\mathscr{C}(u, v)$ takes place at $w_{ij} = u_i + v_j = \ln(\bar{x}_{ij}/M_{ij}), (i, j) \in P$.*

4. UNIQUENESS

Assuming a minumum exists, if p^T, q is a (price, size) solution so is $\bar{p}^T = \alpha p^T, \bar{q} = (\alpha^{-1}) q$ for any $\alpha > 0$. Uniqueness is possible at most to relative prices and sizes. However, extending the argument in Theor. 3, we can show there exists a " basic " solution $\hat{x}_{ij}$ of (3.1) so that

$$(3.6) \qquad \mathscr{C}(u, v) = \sum_{\hat{x}_{ij}>0} (M_{ij} e^{w_{ij}} - \hat{x}_{ij} w_{ij}) + \sum_{\hat{x}_{ij}=0} M_{ij} e^{f_{ij}(w)},$$

where $f_{ij}(w)$ is linear in the independent variables w_{ij}. Evidently the right hand is a strictly convex function of the w_{ij}. (Without assuming a minimum, $\mathscr{C}(u, v)$ may be written in form (3.6) plus possible additional exponential terms in z_{kl}, i.e., as a strictly convex function of w_{ij} and z_{kl}).

THEOREM 4. *The minimum of* $\mathscr{C}(u, v)$ *is unique in variables* w_{ij} *of* (3.6).

THEOREM 5. *If* $\hat{w}_{ij}$ *minimizes* (3.6), *then the set of minimizing* $(\hat{u}, \hat{v})$ *is the set of all solutions to*

$$\begin{aligned} u_i + v_j &= \hat{w}_{ij}, \ \hat{x}_{ij} > 0 \\ u_i + v_j &= f_{ij}(\hat{w}_{ij}), \ \hat{x}_{ij} = 0, \end{aligned} \tag{3.7}$$

where the submatrix of D associated with the $\hat{x}_{ij}$ is a basis for (3.1).

Questions of uniqueness and stability of the system to arbitrary but small perturbations of the γ_i, δ_j are closely related. Stability of solution to such perturbations implies that the rank of D must be $r + s - 1$, i.e.

THEOREM 6. *If no subsums of the* γ_i *and* δ_j *are-equal, then for a minimum to exist the rank of* D *must be* $r + s - 1$. *The minimum point is unique up to relative prices and sizes.*

Sometimes M may be decomposable in block diagonal on suitable interchanges of rows and of columns. If there are B blocks, the problem splits into B independent problems of the same form. When soluble, each independent problem minimum can be at most unique up to a scale factor for prices (and its reciprocal for sizes). It must be so unique if the block problem is stable under local perturbations of its own γ_i, δ_j.

5. COMPUTATION

The accounting balance prices and sizes and corresponding consumer goods prices and employment levels may be computed from (2.6) and its variants by any of the standard convex programming methods, e.g. SUMT or piecewise linearization. Alternately, Newton-Raphson schemes can be applied to solve the balance equations directly, now that one knows the theory of their ambient system. A third direction, which may be most convenient for large systems is through solving the geometric programming dual to (2.5).

From [3], page 210 et seq., the extended geometric program

$$\text{minimize} \quad p^{\mathrm{T}} \mathrm{M} q + \ln [p_1^{-\gamma_1} \cdots p_r^{-\gamma_r} q_1^{-\delta_1} \cdots q_s^{-\delta_s}] \quad \text{with} \quad p^{\mathrm{T}}, q > 0, \tag{5.1}$$

has a dual

$$\text{minimize} \ -v(\delta) \equiv \sum_{(i,j)\in \mathrm{Q}} \delta_{ij} [\ln(\delta_{ij}/e) - \ln \mathrm{M}_{ij}] \text{ subject to (4.1) on the } \delta_{ij}. \tag{5.2}$$

If δ_{ij}^* is an optimal solution, then

$$\delta_{ij}^* = \mathrm{M}_{ij}\, p_i^*\, q_i^* \quad , \quad (i, j) \in \mathrm{Q}. \tag{5.3}$$

Note how (5.3) recovers the result in (3.7).

Since linear programming problems of distribution type may be computed at least 100 times faster than general linear programs, e.g. in a few seconds on existing codes for r, $s \sim 10^2$ (and larger), and since existing codes for distribution problems handle by fai the largest linear programs, this mode of solution seems most attractive for large systems.

6. EXTENSIONS

The model needs extension in the directions of more explicit connections between employment levels, consumer-group sizes, etc. Multi-period extensions would also be desirable, as would introduction of stochastic (risk or uncertainty) elements. The Authors are currently making such extensions.

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