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Characteristic classes of S^1-pseudo-actions in fiber bundles. Nota I

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Topologia algebrica. — *Characteristic classes of* S^1*-pseudo-actions in fiber bundles* (*). Nota I di NICOLAE TELEMAN, presentata (**) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Continuando lo studio delle «classi caratteristiche t_i» già introdotte in [4], si ricostruiscono qui «classi caratteristiche $\mathcal{C}_i$» per involuzioni provenienti da una pseudo-azione del gruppo S^1. L'attuale procedimento appare come un raffinamento di quello di cui in [4].

§ 1. INTRODUCTION

In a recent paper [4] we defined the characteristic classes $t_i(\xi)$ of fiber bundles ξ with involution A. The classes $t_i(\xi)$ generalize the Stiefel-Whitney classes.

It is known that the Stiefel-Whitney classes measure the obstruction to the construction of linear independent sections in a real vector bundle, while the Chern classes measure the obstruction to the construction of **C**–linear independent sections in a complex vector bundle.

If $s_1, \cdots, s_n$ are **C**–linear independent sections in a complex vector bundle ξ, then obviously, $s_1, is_1, \cdots, s_n, is_n$ are **R**-linear independent sections in ξ.

The classes t_i defined in [4] measure the obstruction to the construction of certain operators $k_p^{(r)}$; the operators $k_p^{(r)}$ can be constructed (when the involution is the antipodal map in an Euclidean vector bundle) if we know certain **R**–linear independent sections. The cochains $\omega_r(\xi, k_p^{(r)})$ which measure the obstruction to the existence of the operators $k_p^{(r)}$, are not always cocyles (over an arbitrary ring), and if $d\omega_r(\xi, k_p^{(r)}) = 0$, the cohomology class of the cocycle $\omega_r(\xi, k_p^{(r)})$ depends, in general, of the choice of the operators $k_p^{(r)}$; only a certain reduction of the coefficients assures to be cocyle and the independence of $[\omega_r]$ of $k_p^{(r)}$. This phenomenon occurs because the existence of the involution A alone limits too little the operators $k_p^{(r)}$. In the complex case, the complex structure limits the **R**–linear independent sections as it has been shown above. Therefore, is natural to expect that a certain limitation of the operators $k_p^{(r)}$ can offer a new information. We show that if $k_{p+1}^{(1)}(A+1)\, k_p^{(1)} = 0$, $0 \le p < \infty$, then we can define $k_p^{(2r+1)} = k_{p+2r}^{(1)}\, k_p^{(2r)}$ and we obtain a new system of characteristic classes $\mathcal{C}_i$ which generalize the Chern classes, and whose fit reduction gives t_i. There are geometric conditions, weaker than the S^1–actions, (the associativity does not matter; we call them S^1-pseudo-actions, see § 2) which induce such $k_p^{(1)}$. A S^1-pseudo-action induces an homo-

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topy between I and A; Euclidean sphere bundles with an homotopy between I and the antipodal map have been studied by I. M. James and D. W. Anderson in [1], [2], [3].

§ 2. S^1-PSEUDO-ACTIONS IN FIBER BUNDLES

The motivation of the following notion is contained in the following Proposition 2.4. and the Definition 2.6.

2.1. DEFINITION. *Let be* $\xi = (E, \pi, B, F, A) \in \mathscr{B}(B, F, \mathbf{Z}_2)$ (see 2.4. [4]) *and let* $S_p : C_p(E, R) \to C_{p+1}(E, R)$, $0 \leq p < \infty$, *be local homomorphisms (see the proof of the Proposition 3.3* [4]) *such that*:

$$1 - A = \partial_{p+1} S_p + S_{p-1} \partial_p \quad , \quad S_{p+1}(1 + A) S_p = 0, \qquad 0 \leq p < \infty.$$

Then (ξ, S) will be called a " fiber bundle with homological S^1-pseudo-action S ", and we shall write $(\xi, S) = (E, \pi, B, F, A, S) \in \mathscr{B}(B, F, \mathbf{Z}_2, HS^1)$. In a similar manner we define $\mathscr{B}^r_R(B, F, \mathbf{Z}_2, HS^1)$ (see 7.7. [4]).

2.2. *Remark.* If $(\xi, S) \in \mathscr{B}(B, F, \mathbf{Z}_2, HS^1)$, then $A_* : H_*(E_x, R) \hookleftarrow$, $x \in B$, is the identity.

2.3. *Example.* If $\xi = (E, \pi, B, F, A) \in \mathscr{B}(B, F, \mathbf{Z}_2)$ and if there exists the local chain homotopy $S_p, 0 \leq p < \infty$, such that $S_p A = A S_p$, $S_{p+1} S_p = 0$, then $(\xi, S) \in \mathscr{B}(B, F, \mathbf{Z}_2, HS^1)$.

2.4. PROPOSITION. *Let be* $\xi = (E, \pi, B, F, A)$ *and let* $h_t : E \to E$, $0 \leq t \leq 1$, *be a continuous fiber-preserving homotopy*, $h_0 = 1$, $h_1 = A$, *such that*

$$h_s h_t = h_t h_s \quad , \quad 0 \leq s, t \leq 1.$$

Then the homotopy S *defines a local chain homotopy* S *such that* $(\xi, S) \in \mathscr{B}(B, F, \mathbf{Z}_2, HS^1)$.

Proof. We shall prove that the conditions from the Example 2.3. are fulfilled.

Let $\Delta^r = [x_0, x_1, \cdots, x_r]$ be the standard r-simplex and let $h_r(\Delta^r)$ be the integral chain in the simplicial complex $\Delta^r \times I$, $I = [0, 1]$:

$$h_r(\Delta^r) = \sum_{0 \leq p \leq r} (-1)^p [(x_0, 0), \cdots, (x_p, 0), (x_p, 1), \cdots, (x_p, 1)];$$

evidently, $(\partial h_r + h_{r-1} \partial)(\Delta^r) = \Delta^r \times [0] - \Delta^r \times [1]$.

The chain $h_r(\Delta^r)$ can be considered also as a singular chain, if we agree to identify any simplex σ from $h_r(\Delta^r)$ with the unique linear map which carries the vertices of Δ^{r+1} on the vertices of σ, in the same order.

Let $h : E \times I \to E$ be the function $(e, t) \mapsto h_t e$. It is classically that the correspondence $\sigma \mapsto S_r \sigma$, where $\sigma : \Delta^r \to E$ is a singular simplex, and $S_r \sigma$

is the image of $h_r(\Delta^r)$ by the composition of the maps

$$\Delta^r \times I \xrightarrow{\sigma\times 1} E\times I \xrightarrow{h} E\,, \qquad 0 \leq r < \infty,$$

is a chain homotopy between 1 end A.

We observe that $S_{r+1} S_r(\sigma)$ is the image of the chain $h_{r+1} h_r(\Delta^r)$ by the composition of the maps

$$\Delta^r \times I\times I \xrightarrow{\sigma\times 1\times 1} E\times I\times I \xrightarrow{h\times 1} E\times I \xrightarrow{h} E\,;$$

we observe also that $h_{r+1} h_r(\Delta^r) = a + b - c - d$, where

$$a = \sum_{0\leq p<q, 0<q\leq r} (-1)^{p+q} [(x_0\,,0\,,0)\,,\cdots,(x_p\,,0\,,0)\,,(x_p\,,0\,,1)\,,\cdots$$
$$\cdots,(x_q\,,0\,,1)\,,(x_q\,,1\,,1)\,,\cdots,(x_r\,,1\,,1)].$$

$$b = \sum_{0\leq q\leq r} [(x_0\,,0\,,0)\,,\cdots,(x_q\,,0\,,0)\,,(x_q\,,0\,,1)\,,(x_q\,,1\,,1)\,,\cdots,(x_r\,,1\,,1)],$$

$$c = \sum_{0\leq q\leq r} [(x_0\,,0\,,0)\,,\cdots,(x_q\,,0\,,0)\,,(x_q\,,1\,,0)\,,(x_q\,,1\,,1)\,,\cdots,(x_r\,,1\,,1)],$$

$$d = \sum_{q<p\leq r, 0\leq q<r} (-1)^{p+q} (x_0\,,0\,,0)\,,\ldots,(x_q\,,0\,,0)\,,(x_q\,,1\,,0)\,,\cdots$$
$$\cdots,(x_p\,,1\,,0)\,,(x_p\,,1\,,1)\,,\cdots,(x_r\,,1\,,1)].$$

Let $\Lambda : I\times I$ be the involution $(s\,,t)\mapsto(t\,,s)$. The commutativity $h_s h_t = h_t h_s$ implies the commutativity of the diagram:

$$\begin{array}{ccccccc} \Delta^r\times I\times I & \xrightarrow{\sigma\times 1\times 1} & E\times I\times I & \xrightarrow{h\times 1} & E\times I & \searrow^{h} & \\ \downarrow{\scriptstyle 1\times\Lambda} & & \downarrow{\scriptstyle 1\times\Lambda} & & & & E. \\ \Delta^r\times I\times I & \xrightarrow{\sigma\times 1\times 1} & E\times I\times I & \xrightarrow{h\times 1} & E\times I & \nearrow_{h} & \end{array}$$

We have $(1_{\Delta^r}\times\Lambda)(a) = d$, $(1_{\Delta^r}\times\Lambda)(b) = c$; in consequence

$$S_{r+1} S_r(\sigma) = h\circ(h\times 1_I)\circ(\sigma\times 1_I\times 1_I)(a+b-c-d) =$$
$$= h\circ(h\times 1_I)\circ(\sigma\times 1_I\times 1_I)(1\times\Lambda)(a+b) -$$
$$- h\circ(h\times 1_I)\circ(\sigma\times 1_I\times 1_I)(c+d) = 0,$$

and the conditions from 2.3. are verified.

2.5 *Remark*. h being the homotopy from the Proposition 2.4., we observe that we can define

$$H : E\times S^1 \to E,\ (S^1 = \{\zeta \mid \zeta\in \mathbf{C}\,, |\zeta| = 1\}),$$

$$H(e\,,\zeta) = H_\zeta(e) = \begin{cases} h_t\, e, & \text{for } \zeta = e^{\pi i t}\,, 0\leq t\leq 1, \\ A h_t\, e, & \text{for } \zeta = e^{\pi i(t+1)}. \end{cases}$$

We have also $H_\zeta H_{\zeta'}\,, = H_{\zeta'} H_\zeta\,, AH_\zeta = H_\zeta A$.

2.6. DEFINITION. *If* $H_\zeta : E \to E$, $\zeta \in S^1$, *is a continuous family of fiber-preserving maps such that* $H_1 = 1_E$, $H_{-1} \circ H_{-1} = 1_E$, $H_\zeta \circ H_{\zeta'} = H_{\zeta'} \circ H_\zeta$, *for any* $\zeta, \zeta' \in S^1$, *then* H_ζ *is called a "*S^1*-pseudo-action" on* E. *If we take* $A = H_{-1}$, *then the restriction of* H *to* $S^1_+ = \{\zeta \mid \zeta \in S^1, \operatorname{Im} \zeta \geq 0\}$ *defines a homological* S^1*-pseudo-action as has been proved in Proposition 2.4..*

2.7. *Example.* A S^1-action is a S^1 pseudo-action. In particular, a complex vector bundle and his associate sphere bundle (by respect an Hermitian metric) are bundles with S^1-pseudo-action.

2.8. PROPOSITION. *Let* H_ζ *be a* S^1*-pseudo-action. If* $\rho_n : S^1 \to S^1$ *is the map* $\zeta \mapsto \zeta^{2n-1}$, *then* $(\rho_n H)_\zeta = H_{\zeta^{2n+1}}$ *is also a* S^1*-pseudo-action. Also* $(\bar{H})_\zeta = H_{\bar{\zeta}}$ *is a* S^1*-pseudo-action.*

Proof. Obvious.

2.9. *Remark.* The S^1-pseudo-actions H, $\rho_n H$, $\bar{H}$ induce different homological S^1-pseudo-actions S, $\rho_n S$, $\bar{S}$ though the three chain homotopies satisfy the same equations

$$1 - A = \partial k + k\partial, \quad k = S, \quad \text{resp. } \rho_n S, \quad \text{resp. } \bar{S}$$

$$A = H_{-1} = (\rho_n H)_{-1} = \bar{H}_{-1}.$$

2.10. DEFINITION. *If* $\xi_i \in \mathscr{B}(E_i, \pi_i, B, F_i, H_{\zeta,i}) \in \mathscr{B}(B, F_i, S^1)$, $i = 1, 2$, *then in* $\xi_1 \oplus \xi_2$ *(see 2.11.* [4]) *we define the* S^1*-pseudo-action* H_ζ: *for*

$$e = (e_1, t, e_2) \in \{e_1\} * \{e_2\} \quad , \quad e_i \in E_{i,x} \quad , \quad x \in B,$$

$$H_\zeta e = (H_{\zeta,1} e_1, t, H_{\zeta,2} e_2).$$

Therefore $\mathscr{B}(B, F_i, S^1)$ is closed by respect the sum $\oplus$. In particular, if $\xi \in \mathscr{B}(B, F, S^1)$, then $\Sigma\Sigma\xi \in \mathscr{B}(B, F, S^1)$ because $(B \times S^1, pr_1, B, S^1, S^1) \in \mathscr{B}(B, S^1, S^1)$.

§ 3. CHARACTERISTIC CLASSES OF HOMOLOGICAL S^1-PSEUDO-ACTIONS

In this paragraph we use the notations from § 7 [4].

3.1. *Theorem. Let be* $(\xi, S) = (E, \pi, B, F, A, S) \in \mathscr{B}^n_R(B, F, Z_2, HS^1)$. *Then*:

(i) *there exists the local* R*-homomorphisms*

$$k_p^{(r)} : C_p(E, R) \to C_{p+r}(E, R), \qquad 0 \leq p + r \leq n,$$

$$k_p^{(0)} = 1 \quad , \quad k_p^{(1)} = S_p \quad , \quad k_p^{(2r+1)} = S k_p^{(2r)},$$

such that

$$(1 + (-1)^r A)\, k_p^{(r-1)} = \partial k_p^{(r)} + (-1)^{r+1} k_{p-1}^{(r)} \partial,$$

(ii) *if* $k_p^{(r)}$, $\tilde{k}_p^{(r)}$ *are two systems of local homomorphisms* (*i*), *then there exists the systems of local* R*–homomorphisms*

$$\tilde{\varphi}_p^{(2r)} : C_p(E , R) \to C_{p+2r+1}(E , R) , \qquad p + 2r + 1 \leq n,$$

such that, if we denote $K_p^{(r)} = k_p^{(r)} - \tilde{k}_p^{(p)}$, *we have*

$$K_p^{(2r)} = -(1 + A)\, S\tilde{\varphi}_p^{(2r-2)} + \partial\tilde{\varphi}_p^{(2r)} + \tilde{\varphi}_{p-1}^{(2r)}\partial, \quad \textit{and}$$

$$K_{n-2r}^{(2r)} = -(1 + A)\, S\tilde{\varphi}_{n-2r}^{(2r-2)} + \tilde{\mu}_{n-2r}^{(2r)} + \tilde{\varphi}_{n-2r-1}^{2r}\,\partial , \qquad \partial\tilde{\mu}_{n-2r}^{(2r)} = 0;$$

$$K_p^{(2r+1)} = SK_p^{(2r)} = (1 - A)\,\tilde{\varphi}_p^{(2r)} - \partial(S\tilde{\varphi}_p^{2r}) + (S\tilde{\varphi}_{p-1}^{(2r)})\,\partial , \qquad p + 2r + 2 \leq n.$$

Proof. We indicate here only the specific part of the proof by respect the proof of the Theorem 7.1. [4].

(i) We suppose we have been constructed $k_p^{(2r)}$.

We define: $k_p^{(2r+1)} = Sk_p^{(2r)}$; we have:

$$\partial k_p^{(2r+1)} + k_p^{(2r+1)}\,\partial = (1 - A - S\partial)\, k_p^{(2r)} + Sk_{p-1}^{(2r)}\,\partial_p =$$
$$= (1 - A)\, k_p^{(2r)} - S(\partial k_p^{(2r)} - k_{p-1}^{(2r)}\,\partial_p) = (1 - A)\, k_p^{(2r)}.$$

(ii) We suppose we have been established $K_p^{(2r+1)} = (1 - A)\,\tilde{\varphi}_p^{(2r)} - \partial(S\tilde{\varphi}_p^{(2r)}) + (S\tilde{\varphi}^{(2r)})\,\partial_p$, $p + 2r + 2 \leq n$. Then we have:

$$(1+A)\, K_p^{(2r+1)} = \partial K_p^{(2r+2)} - K^{(2r+2)}\,\partial_p \quad \text{or}$$

$$\partial K_p^{(2r+2)} - K_p^{(2r+2)}\,\partial_p = -(1 + A)\,(\partial S\tilde{\varphi}_p^{(2r)} - S\tilde{\varphi}^{(2r)}\,\partial_p),$$

which gives

$$\partial(K_p^{(2r+2)} + (1 + A)\, S\tilde{\varphi}_p^{(2r)}) - (K_p^{(2r+2)} + (1 + A)\, S\tilde{\varphi}^{(2r)})\,\partial_p = 0.$$

We obtain by increased induction over p

$$K_p^{(2r+2)} + (1 + A)\, S\tilde{\varphi}_p^{(2r)} = \begin{cases} \partial\tilde{\varphi}_p^{(2r+2)} + \tilde{\varphi}_{p-1}^{(2r+2)}\,\partial_p , & p + 2r + 3 \leq n , \\ \tilde{\mu}_{n-(2r+2)}^{(2r+2)} + \tilde{\varphi}^{(2r+2)}\,\partial_{n-2r-2} , & \partial\tilde{\mu}_{n-(2r+2)}^{(2r+2)} = 0 \end{cases}$$

which proves the assertion.

3.2. COROLLARY. *If we confront the Theorem 3.1. with the Theorem 7.1.* [4], *we have*

$$\varphi_p^{(2r)} = \tilde{\varphi}_p^{(2r)} \quad , \quad p + 2r + 1 \leq n \quad , \quad \mu_{n-2r}^{(2r)} = \tilde{\mu}_{n-2r}^{(2r)} ,$$
$$\varphi_p^{(2r+1)} = -S\tilde{\varphi}_p^{(2r)} \quad , \quad p + 2r + 2 \leq n \quad , \quad \mu_{n-2r-1}^{(2r+1)} = -\partial(S\tilde{\varphi}_{n-2r-1}^{(2r)}).$$

3.3. PROPOSITION. *With the notations above, for the upper special operators* $k_p^{(r)}$, *we have*:

(i) $\omega_{2r+1}(\xi , k_p^{(r)}) = 0$

(ii) $d\omega_{2r}(\xi , k_p^{(r)}) = 0.$

Proof. (i) Using the Corollary 3.2., we have (see 7.2. [4]):

$$\omega_{2r+1}(\xi, k_p^{(r)})(\sigma) = [((1-A)\,k_{n-2r}^{(2r)} - k_{n-(2r+1)}^{(2r+1)}\,\partial)(\sigma)] = [\partial k_{n-2r}^{(2r+1)}(\sigma)] = 0$$

(σ being an arbitrary $(n-2r)$-singular simplex), because the operators $k_p^{(2r+1)} = S k_p^{(2r)}$ exist for $p + 2r \leq n$).

(ii) The general relation (2) from 7.2. [4] gives

$$d\omega_{2r}(\xi, k_p^{(r)}) = -2\,\omega_{2r-1}(\xi, k_p^{(r)}) = 0.$$

3.4. PROPOSITION. *If* $k_p^{(r)}$, $\tilde{k}_p^{(r)}$ *are two systems of homomorphisms* (i) *Theorem 3.1., then*

$$\omega_{2r}(\xi, k_p^{(r)}) - \omega_{2r}(\xi, \tilde{k}_p^{(r)}) = d[\mu_{n-2r}^{(2r)}].$$

Proof. From the proof of the Theorem 7.5. [4] and the Corollary 3.2., we know (σ being a singular simplex):

$$(\omega_{2r}(\xi, k_p^{(r)}) - \omega_{2r}(\xi, \tilde{k}_p^{(r)}))(\sigma)) = (1 + A_*)[\mu_{n-2r+1}^{(2r-1)}(\sigma)] +$$
$$+ d[\mu_{n-2r}^{(2r)}(\sigma)] = 2[-\partial S_{n-2r+1}^{(2r-2)}(\sigma)] + (d[\mu_{n-2r}^{(2r)}])(\sigma) = (d[\mu_{n-2r}^{(2r)}])(\sigma).$$

If we gather the content of the Propositions 3.3., 3.4., we have the

3.5. THEOREM. *If* $(\xi, S) \in \mathscr{B}_R^n(B, F, Z_2, HS^1)$, *then* $\omega_{2r+1}(\xi, k_p^{(r)}) = 0$, *and* $\omega_{2r}(\xi, k_p^{(r)})$ *individuates a well defined cohomology class*

$$[\omega_{2r}(\xi, k_p^{(r)})] \in \pi^* \mathscr{H}^{n-2r+1}(B, \mathscr{H}_n(\xi, R))$$

for arbitrary $k_p^{(r)}$ *as in the Theorem 3.1. (i).*

3.6. DEFINITION. If $(\xi, S) \in \mathscr{B}_R^n(B, F, Z_2, HS^1)$ we define

$$\mathscr{C}_i(\xi, S) = s^* \omega_{n-2i+1}(\Sigma\Sigma\xi, k_p^{(r)}), \qquad 2i \leq n+1,$$

$s: B \to \Sigma\Sigma\xi$ being a section (see 7.9. [4]), and $k_p^{(r)}$ being a system of homomorphisms as in the Theorem 3.1. (i) for $\Sigma\Sigma\xi$ which is also a fiber bundle with homological S^1-pseudo-action.

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