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A note on Multi-valued contraction mappings

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Analisi funzionale. — *A note on Multi-valued contraction mappings.*
Nota di Barada K. Ray, presentata [(*)] dal Socio E. Bompiani.

Riassunto. — Vengono dimostrati in questa Nota due nuovi teoremi sulla esistenza di un punto fisso di una corrispondenza, che sia plurivalente ed abbia carattere di contrazione, di uno spazio metrico completo in sé.

Introduction

The classical Banach's contraction principle states that if (X, ρ) [(1)] is a complete metric space and if T is a contraction mapping $\{$i.e. $\rho(Tx, Ty) \leq \leq \alpha\rho(x, y)$ for all $x, y \in X, 0 \leq \alpha < 1\}$ of X into itself then there is a unique fixed point of T. A similar theorem (stated below) of multivalued contraction mappings has been given by Nadler [2].

Theorem 2. *Let* (X, ρ) *be a complete metric space. If* $F: X \to CB(X)$ *is a multivalued contraction mapping then* F *has a fixed point.*

Recently [1] we have been able to prove the following theorem.

Theorem 1. *If* T *is a (single valued) mapping of a complete metric space* (X, ρ) *into itself and if*

$$\rho(Tx, Ty) \leq \alpha[\rho(x, Tx) + \rho(y, Ty) + \rho(x, y)] \quad \text{all } x, y \in X, \quad 0 < \alpha < 1/3$$

then T *has a unique fixed point.*

The aim of this Note is to extend the above result to multivalued mappings and to prove a related theorem on fixed point.

Preliminaries

We use the notations and definitions as given in [2]

Definition 1. *If* (X, ρ) *is a metric space then*:

(i) $CB(X) = \{C \mid C$ *is a non-empty closed and bounded subset of* $X\}$;

(ii) $\delta(x, A) = \inf\{\rho(x, y) : y \in A\}$

(iii) $N(\varepsilon, C) = \{x \in X \mid \rho(x, c) < \varepsilon$ *for some* $c \in C\}$ *where* $\varepsilon > 0$ *and* $C \in CB(X)$;

(iv) $H(A, B) = \inf\{\varepsilon \mid A \subset N(\varepsilon, B)$ *and* $B \subset N(\varepsilon, A)\}$, $\varepsilon > 0$ *and* $A, B \in CB(X)$.

(*) Nella seduta del 20 aprile 1974.

(1) For the meaning of the symbols see Nadler [2].

The function H is a metric for CB (X), called the Hausdorff metric for CB (X)

DEFINITION 2. *Let* (X, ρ_1) *and* (Y, ρ_2) *be two metric spaces. A function* $F : X \to CB(Y)$ *is said to be a multivalued contraction mapping of* X *into* Y *if*

$$H(F(x'), F(y')) \leq \alpha \rho(x', y') \quad , \quad 0 \leq \alpha < 1$$

for all $x', y' \in X$

DEFINITION 3. *A point* x *is said to be a fixed point of a multivalued mapping* F *if* $x \in F(x)$.

THEOREM 1. *Let* (X, ρ) *be a complete metric space.*
If $F : X \to CB(X)$ *be a multivalued mapping satisfying.*

$$H(F(x), F(y)) \leq \alpha [\delta(x, F(x)) + \delta(y, F(y)) + \rho(x, y)],$$
$$0 \leq \alpha < 1/3 \quad , \quad x, y \in X$$

and if F *be continuous on* X *then* F *has a fixed point.*

Proof. Let $x_0 \in X$, then $F(x_0) \in CB(X)$. Pick a point $x_1 \in F(x_0)$.
Since $F(x_0), F(x_1) \in CB(X)$ and $x_1 \in F(x_0)$ there is a point $x_2 \in F(x_1)$ such that

$$\rho(x_1, x_2) \leq H(F(x_0), F(x_1)) + \frac{2\alpha}{1-\alpha} \; ^{***},$$

Similarly, since $F(x_1), F(x_2) \in CB(X)$, there exists a point $x_3 \in F(x_2)$ such that

$$\rho(x_2, x_3) \leq H(F(x_1), F(x_2)) + \left(\frac{2\alpha}{1-\alpha}\right)^2.$$

Proceeding in this manner we get a sequence $\{x_i\}_{i=1}^{\infty}$ of points in X such that $x_i \in F(x_{i-1})$ and that

$$\rho(x_i, x_{i+1}) \leq H(F(x_{i-1}), F(x_i)) + \left(\frac{2\alpha}{1-\alpha}\right)^i.$$

Now

$$\delta(x_i, F(x_i)) = \inf\{\rho(x_i, z) : z \in F(x_i)\} \qquad \text{for all } i$$
$$\leq \rho(x_i, x_{i+1}), \qquad \text{since } x_{i+1} \in F(x_i).$$

Now

$$\rho(x_i, x_{i+1}) \leq H(F(x_{i-1}), F(x_i)) + \left(\frac{2\alpha}{1-\alpha}\right)^i$$
$$\leq \alpha [\delta(x_{i-1}, F(x_{i-1})) + \delta(x_i, F(x_i)) + \rho(x_{i-1}, x_i)] + \left(\frac{2\alpha}{1-\alpha}\right)^i.$$

Therefore

$$\rho(x_i, x_{i+1}) \leq \alpha\rho(x_{i-1}, x_i) + \alpha\rho(x_i, x_{i+1}) + \alpha\rho(x_{i-1}, x_i) + \left(\frac{2\alpha}{1-\alpha}\right)^i.$$

Hence

$$\rho(x_i, x_{i+1}) \leq \frac{2\alpha}{1-\alpha}\, \rho(x_{i-1}, x_i) + \frac{(2\alpha)^i}{(1-\alpha)^{i+1}}\,.$$

Again

$$\rho(x_{i-1}, x_i) \leq \frac{2\alpha}{1-\alpha}\, \rho(x_{i-2}, x_{i-1}) + \frac{(2\alpha)^{i-1}}{(1-\alpha)^i}\,.$$

So

$$\begin{aligned}\rho(x_i, x_{i+1}) &\leq \left(\frac{2\alpha}{1-\alpha}\right)^2 \rho(x_{i-2}, x_{i-1}) + \frac{(2\alpha)^i}{(1-\alpha)^{i+1}} + \frac{(2\alpha)^i}{(1-\alpha)^{i+1}} \\ &= \left(\frac{2\alpha}{1-\alpha}\right)^2 \rho(x_{i-2}, x_{i-1}) + \frac{2}{2\alpha}\left(\frac{2\alpha}{1-\alpha}\right)^{i+1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq \left(\frac{2\alpha}{1-\alpha}\right)^i \rho(x_0, x_1) + \frac{i}{2\alpha}\left(\frac{2\alpha}{1-\alpha}\right)^{i+1}.\end{aligned}$$

Now

$$\begin{aligned}&\rho(x_i, x_{i+k}) \\ &\leq \rho(x_i, x_{i+1}) + \rho(x_{i+1}\, x_{i+2}) + \cdots + \rho(x_{i+k-1}, x_{i+k}) \\ &\leq \left(\frac{2\alpha}{1-\alpha}\right)^i \rho(x_0, x_1) + \frac{i}{2\alpha}\left(\frac{2\alpha}{1-\alpha}\right)^{i+1} \\ &+ \left(\frac{2\alpha}{1-\alpha}\right)^{i+1} \rho(x_0, x_1) + \frac{i+1}{2\alpha}\left(\frac{2\alpha}{1-\alpha}\right)^{i+1} + \cdots \\ &\cdots + \left(\frac{2\alpha}{1-\alpha}\right)^{i+k-1} \rho(x_0, x_1) + \frac{i+k-1}{2\alpha}\left(\frac{2\alpha}{1-\alpha}\right)^{i+1} \\ &= \sum_{n=i}^{i+k-1}\left(\frac{2\alpha}{1-\alpha}\right)^n \rho(x_0, x_1) + \frac{1}{2\alpha}\sum_{n=i}^{i+k-1} n\left(\frac{2\alpha}{1-\alpha}\right)^{n+1}, \quad n, k > 1.\end{aligned}$$

Now

$$\frac{2\alpha}{1-\alpha} < 1 \qquad \text{since} \quad \alpha < \frac{1}{3}\,.$$

So $\rho(x_i, x_{i+k})$ becomes sufficiently small as $i \to \infty$.

Hence the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence.

Since (X, ρ) is a complete metric space, we infer that $\lim_{i\to\infty} x_i = z_0 \in X$, and since F is continuous $\{F(x_i)\}_{i=1}^{\infty}$ converges to $F(z_0)$.

Now

$$x_i \in F(x_{i-1}) \qquad \text{for all} \quad i = 1, 2, \cdots.$$

Hence $z_0 \in F(z_0)$. So z_0 is a fixed point of F. This completes the proof.

If *** $\quad A, B \in CB(X), \quad \bar{x} \in A \quad$ and $\lambda > 0$

then from the definition of $H(A, B)$ we infer that these exists a $\bar{y} \in B$ such that $\rho(\bar{x}, \bar{y}) \leq H(A, B) + \lambda$.

In the proof of the above theorem $\frac{2\alpha}{1-\alpha}$ and consequently $\left(\frac{2\alpha}{1-\alpha}\right)^i$ play the role of λ, $i = 1, 2, \cdots$

THEOREM 2. *Let* (X, ρ) *be a metric space and* $F_i : X \to CB(X)$ *be a multivalued contraction mapping with fixed point* x_i *for each* $i = 1, 2, \cdots$ *and let* $F_0 : X \to CB(X)$ *be a multivalued contraction mapping. If the sequence* $\{F_i\}_{i=1}^{\infty}$ *converges pointwise to* F_0 *and if* $\lim_{i\to\infty} x_i = x_0$ *then* x_0 *is a fixed point of* F_0

Proof. Let $\varepsilon > 0$, then by the given condition there is an integer N such that

$$H(F_i(x_0), F_0(x_0)) < \frac{\varepsilon}{2} \quad \text{and} \quad \rho(x_i, x_0) < \frac{\varepsilon}{2} \quad \text{for all} \quad i > N.$$

so

$$\begin{aligned} & H(F_i(x_i), F_0(x_0)) \\ & \leq H(F_i(x_i), F_i(x_0)) + H(F_i(x_0), F(x_0)) \\ & < \rho(x_i, x_0) + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus

$$\lim_{i\to\infty} F_i(x_i) = F_0(x_0).$$

Now since $x_i \in F_i(x_i)$ for all $i = 1, 2, \cdots$ it follows that $x_0 \in F(x_0)$.
This completes the proof.

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