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On a generalization of Riesz operators, I

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Analisi funzionale. — *On a generalization of Riesz operators*, I. Nota di GHEORGHE CONSTANTIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore introduce una nuova classe di operatori $\tilde{\mathscr{R}}$ che contiene gli operatori con parte immaginaria compatta e gli operatori di Riesz e ne dà alcune proprietà spettrali.

1. In spectral theory of non- self-adjoint operators on Hilbert space the class of operators whose imaginary parts are compact operators $\left(\frac{T-T^*}{2i} = \text{compact}\right)$ has certain features of great interest. M. S. Brodskii and M. S. Livsic [1], [5] have developed a theory of the triangular form of operators whose imaginary parts belong to the trace class. A generalization of this theory has been obtained by J. T. Schwartz [6]. It is also known that many examples of operators in a Hilbert space H which are spectral in the sense of Dunford are constructed by addition of a compact operator and a self-adjoint one.

It is the aim of this Note to introduce a new class of operators which generalizes the class of operators with compact imaginary part and to give some properties. The definition and some properties make sense also for operators on Banach spaces; however we consider only Hilbert space operators.

2. Let T a bounded linear operator on a Hilbert space H. If M , N are closed subspaces of H invariant under T such that H is the direct sum $M \oplus N$ then the pair (M , N) is said to reduce T. The range and null space of operator T will be denoted by $R\{T\}$, $N\{T\}$ respectively.

A Riesz point of $\sigma(T)$ is a point $\lambda \in \sigma(T)$ such that

$$H = N(\lambda ; T) \oplus F(\lambda ; T)$$

where $\dim N(\lambda ; T) < \infty$ and $F(\lambda ; T)$ is closed, the pair (N , F) reduces T, and $T - \lambda I$ restricted to $N(\ ; T)$ is nilpotent while $T - \lambda I$ restricted to $F(\lambda ; T)$ is a homeomorphism.

A bounded linear operator T is a Riesz operator in H if each point of $\sigma(T) - \{0\}$ is a Riesz point.

DEFINITION 2.1. *An operator* $T \in \mathscr{L}(N)$ *is said to be of* $\tilde{\mathscr{R}}$ *class if every point* $\lambda \in \sigma(T)$. $\operatorname{Im} \lambda \neq 0$ *is a Riesz point of* $\sigma(T)$.

It is clear that every Riesz operator is an operator of class $\tilde{\mathscr{R}}$.

In what follows we give some properties of operators $T \in \tilde{\mathscr{R}}$.

(*) Nella seduta del 20 aprile 1974.

THEOREM 2.1. *Let* $T \in \tilde{\mathcal{R}}$, $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda \neq 0$ *and* $\{x_n\}, \{y_n\}$ *two sequences with the property*

$$(T - \lambda I) x_n = y_n \qquad (n = 1, 2, 3, \ldots).$$

If $\{y_n\}$ *is convergent and* $\{x_n\}$ *is a bounded sequence then there exists a convergent subsequence* $\{x_{n_k}\}$.

If λ *is not a proper value for* T *then* $\{x_n\}$ *is a convergent sequence.*

Proof. Since λ is a Riesz point of $\sigma(T)$ we have

$$H = N(\lambda; T) \oplus F(\lambda; T)$$

hence

$$x_n = x_n^N + x_n^F$$

and since $N(\lambda; T)$, $F(\lambda; T)$ are closed subspaces, it follows that the associated projections P_N, P_F of H onto N an F are continuous [7, p. 243] and therefore

$$\| x_n^N \| = \| P_N x_n \| \le \| P_N \| \| x_n \| \le k$$

from which we conclude that $\{x_n^N\}$ contains a convergent subsequence $\{x_{n_k}^N\}$.

But

$$(T - \lambda I) x_n^F = y_n - (T - \lambda I) x_n^N = y_n^F$$

and by the fact that $\{y_{n_k}\}$ is a convergent sequence it follows that there exists a subsequence $\{y_{n_{k'}}\}$ with the property that $\{y_{n_{k'}}^F\}$ is a convergent subsequence. Then

$$(T - \lambda I) x_{n_{k'}}^F = y_{n_{k'}}^F$$

and therefore $\{x_{n_{k'}}^F\}$ is a convergent subsequence (since $T - \lambda I$ is a homeomorphism) and we obtain that

$$x_{n_{k'}} = x_{n_{k'}}^N + x_{n_{k'}}^F$$

which is a convergent subsequence of $\{x_n\}$.

For the rest part of proposition we observe that if $\{x_n\}$ has two convergent subsequences respectively to $x_0, x'_0, x_0 \neq x'_0$ then

$$(T - \lambda I) x_0 = (T - \lambda I) x'_0 = \lim_{n \to \infty} y_n$$

from which it follows that λ is a proper value for T and the proposition is proved.

If $T \in \tilde{\mathcal{R}}$ then T has the following properties.

1) every point $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda \neq 0$ is an isolated eigenvalue of $\sigma(T)$ and for such a λ if $\gamma(\lambda)$ is a circle in $\rho(T)$ of centre λ such that λ is the only point of $\sigma(T)$ inside $\gamma(\lambda)$ then

$$P(\lambda; T) = -\frac{1}{2\pi i} \int_{\gamma(\lambda)} (T - \lambda I)^{-1} \, d\lambda$$

is a finite-dimensional projection associated with λ and

$$\{x : (T - \lambda I)^{\nu} x = 0\} = P(\lambda ; T) H = N(\lambda ; T)$$

where ν is the algebraic multiplicity of the eigenvalue λ (hence λ is a pole of order ν for $(T - \lambda I)^{-1}$).

2) the set $\{\lambda : \lambda \in \sigma(T), \operatorname{Im} \lambda \neq 0\}$ is at most countable and a limit point of this set is on real line.

Indeed, let $\varepsilon > 0$ and $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ an infinite sequence of distinct points of $\sigma(T)$ which satisfies the condition $|\operatorname{Im} \lambda_k| > \varepsilon$, $k = 1, 2, \cdots$. Then the sequence $\{\lambda_k\}$ contains a convergent subsequence to a point $\lambda_0 \in \sigma(T)$ with $|\operatorname{Im} \lambda_0| > \varepsilon$ which is a contradiction. Therefore, for each $\varepsilon > 0$ there exists only a finite set of points $\lambda \in \sigma(T)$ with $|\operatorname{Im} \lambda| > \varepsilon$ and the assertion is proved.

3) If $\lambda_0 \in \sigma(T)$, $\operatorname{Im} \lambda_0 \neq 0$ is a Riesz point of $\sigma(T)$ then $\bar{\lambda}_0$ is a Riesz point of $\sigma(T^*)$ with the same algebraic multiplicity.

Using some results contained in [2], [3], [4] it is possible to give another property since it can be proved, as for the operators with compact imaginary part, that $R\{T - \lambda_0 I\}$, $\lambda_0 \in \sigma(T)$, $\operatorname{Im} \lambda_0 \neq 0$, is a closed set. Then for each $\lambda_0 \in \sigma(T)$, $\operatorname{Im} \lambda_0 \neq 0$ there exists $\varepsilon(\lambda_0) > 0$ and a nonnegative integer $r(\lambda_0)$ such that for every λ, $0 < |\lambda - \lambda_0| < \varepsilon(\lambda_0)$ we have

$$n(T - \lambda_0 I - \lambda I) = n(T - \lambda_0 I) - r(\lambda_0)$$

$$d(T - \lambda_0 I - \lambda I) = d(T - \lambda_0 I) - r(\lambda_0)$$

$$n(T - \lambda_0 I - \lambda I) - d(T - \lambda_0 I - \lambda I) = n(T - \lambda_0 I) - d(T - \lambda_0 I)$$

and

$$R\{T - \lambda_0 I - \lambda I) \text{ is closed}$$

where $n(T) = \dim N\{T\}$ and $d(T) = \operatorname{codim} R\{T\}$.

A closed linear operator T is said to be semi-Fredholm if $R\{T\}$ is closed and at least one of $n(T)$ and $d(T)$ is finite. For an operator $T \in \tilde{\mathscr{R}}$ we give the following characterization:

THEOREM 2.2. *An operator* $T \in \tilde{\mathscr{R}}$ *if and only if* $T \in \mathscr{L}(H)$ *and* $T - \lambda I$ *is semi-Fredholm for every* λ *in the complement of the set of all* $\mu \in \sigma(T)$. $\operatorname{Im} \lambda \neq 0$ *and is not a Riesz point for* $\sigma(T)$.

COROLLARY. *If* $T \in \tilde{\mathscr{R}}$ *and* $\dim H = \infty$ *then there exists* $\lambda \in \sigma(T)$, $\operatorname{Im} \lambda = 0$.

Indeed, in the contrary case, $T - \lambda I$ is a semi-Fredholm operator for any $\lambda \in C$ and therefore T is a Riesz operator. Since $0 \in \rho(T)$ it follows that $T^{-1} \in \mathscr{L}(H)$ and hence $I = TT^{-1}$ is a Riesz operator. But then

$$\sigma(I) = \{1\} \qquad \text{and} \quad N(1 ; I) = P(1 ; I) H = H$$

which implies that $\dim H < \infty$ and the corollary follows.

First we need two definitions. A topological space is totally disconnected if its topology has a base consisting of sets simultaneously open and closed. If A and B are two closed subsets of C the Hausdorff distance between A and B is defined as

$$\delta(A, B) = \max\{\sup_{a\in A} \delta(a, B), \sup_{b\in B} \delta(b, A)\}$$

where

$$\delta(a, B) = \inf_{b\in B} |a - b|.$$

If we denote by $\sigma_i(T) = \{\lambda \in \sigma(T), \operatorname{Im}\lambda \neq 0\}$ and $P_j = P(\lambda_j; T)$, $\operatorname{Im}\lambda_j \neq 0$, then we have

LEMMA 2.1. *Let* $T \in \tilde{\mathscr{R}}$ *and put* $P(n) = \sum_{j=1}^{n} P_j$, $C_n = TP(n)$ *and* $S_n = T - C_n$. *Then* $C_n S_n = S_n C_n = 0$, $\sigma_i(C_n) = \{\lambda_j\}_{j=1}^{n}$ *and* $\sigma_i(S_n) = \sigma_i(T) - \sigma_i(C_n)$.

Proof. Since $\{\lambda_j\}_{j=1}^{n}$ is an isolated part of $\sigma(T)$ and $P(n)$ is the projection associated with $\{\lambda_j\}_{j=1}^{n}$ then the pair $(R\{P(n)\}, N\{P(n)\})$ reduce T and

$$\sigma(C_n) = \sigma(C_n|_{R\{P(n)\}}) \cup \sigma(C_n|_{N\{P(n)\}}) = \{\lambda_j\}_{j=1}^{n} \cup \{0\}.$$

Hence $\sigma_i(C_n) = \{\lambda_j\}_{j=1}^{n}$ and the result for $\sigma_i(S_n)$ follows as for $\sigma_i(C_n)$. From the fact that $I - P(n)$ is a projection and $S_n = T(I - P(n))$ we obtain that

$$C_n S_n = S_n C_n = 0$$

Now we shall need the following result due to Newburg.

LEMMA 2.2. *Let* $\mathscr{A}$ *be a Banach algebra and let* $\{T_n\}$ *be a sequence in* $\mathscr{A}$ *converging to* T. *If* $\sigma(T)$ *is a totally disconnected subset of* C *then* $\sigma(T_n) \to \sigma(T)$ *in the Hausdorff metric.*

PROPOSITION 2.1. *If* $\sum_{j=1}^{\infty} TP_j$ *is convergent for some arrangement of the* λ_j*'s in the uniform topology and* $T \in \tilde{\mathscr{R}}$ *with* $\sigma(T)$ *a totally disconnected set, then*

$$T = C + S$$

where C *is a compact operator,* $CS = SC = 0$ *and* $\sigma(S)$ *is a subset of real numbers.*

Proof. First we note that for each $\varepsilon > 0$ there is a n such that $|\operatorname{Im}\lambda_j| \leq \varepsilon$ for $j \geq n$. Let $C_n = \sum_{j=1}^{n} TP_j$ and $S_n = T - C_n$, then by Lemma 2.1 we have

$$\sigma(S_n) \subseteq \{z : |\operatorname{Im} z| \leq \varepsilon\}.$$

Since

$$C_n \to C = \sum_{j=1}^{\infty} TP_j \qquad (n \to \infty)$$

and C_n is of finite rank, we obtain that C is a compact operator. Also

$$S_n \to S = T - C \qquad (n \to \infty)$$

and by Lemma 2.1.

$$C_n S_n = S_n C_n = 0$$

which gives

$$CS = SC = 0.$$

By hypothesis $\sigma(T)$ is a totally disconnected set and therefore $\sigma(S_n) \to \sigma(S)$ in the Hausdorff metric and since ε is arbitrary it follows that $\sigma(S)$ is a real set and the theorem is proved.

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