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Oscillation theorems for certain second-order delay equations

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Equazioni differenziali ordinarie. — *Oscillation theorems for certain second-order delay equations.* Nota di B. S. LALLI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà alcuni teoremi di oscillazione per due classi di equazioni differenziali ad argomento ritardato

$$x'' + g(x')\, \mathrm{F}(t, x(t), x(t - \tau(t))) = 0,$$

$$x'' + g(x')\, \mathrm{G}(t, x(t - \tau(t)), x'(t - \tau(t))) = 0,$$

facendo uso di due diseguaglianze differenziali della forma

$$x'' + ka(t)\, \varphi_1(x)\, g(x') \geq 0 \quad , \quad x'' + ka(t)\, \varphi_2(x)\, g(x') \leq 0 .$$

1. INTRODUCTION

In this paper it is shown that the oscillation properties of large classes of equations

$$x'' + g(x')\, \mathrm{F}(t, x(t), x(t - \tau(t))) = 0 \qquad \left(' = \frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{1.1}$$

and

$$x'' + g(x')\, \mathrm{G}(t, x(t - \tau(t)), x'(t - \tau(t))) = 0, \tag{1.2}$$

can be established by using differential inequalities of the form

$$x'' + ka(t)\, \varphi_1(x)\, g(x') \geq 0, \qquad k > 0, \tag{1.3}$$

and

$$x'' + ka(t)\, \varphi_2(x)\, g(x') \leq 0. \tag{1.4}$$

Bradley [1] discussed equation (1.1) by taking $g(x') \equiv 1$ and $\mathrm{F} = a(t)\, x(t - \tau)$. Oscillatory properties of (1.1) have been established by Gollwitzer [6] who took $\mathrm{F} = [x(t - \tau)]^{\gamma}$, where γ is the ratio of two odd nonnegative integers. Our results, in this paper, are generalizations of some of these results including some of those of Teufel [2] and Waltman [5].

We believe that our results for (1.2) are new. The functions $a, g, \varphi_1, \varphi_2, \tau$, F and G are assumed to be real-valued and smooth enough to ensure the existence and uniqueness of the solutions of $(1 \cdot j)$, $j = 1, 2, 3, 4$, on the t-internal $[0, \infty)$.

(*) Nella seduta del 20 aprile 1974.

2. In the sequel it is assumed that

(i) $x\varphi_i(x) > 0$ for $x \neq 0$, $\frac{d\varphi_i}{dx} \geq 0$ $(i = 1, 2)$;

(ii) F is a continuous function of t, u and v, $a(t)\varphi_2(u) \leq F(t, u, u) \leq \leq b(t)\varphi_1(u)$ for all u and v and F is nondecreasing function of u and v for $uv > 0$;

(iii) G is a continuous function of t, u and v, $a(t)\varphi_2(u) \leq G(t, u, v) \leq \leq b(t)\varphi_1(u)$ for all u and v, and G is nondecreasing in u and nonincreasing in v;

(iv) $\sup_t \tau(t) = \tau < \infty$;

(v) $g(v) \geq c > 0$ for $|v| < \infty$ and $a(t)$, $b(t)$ are continuous functions of t.

A solution $x(t)$ of $(1 \cdot j)$ $(j = 1, 2, 3, 4)$ will be said to be ultimately positive (negative) if there exists a $T \in [0, \infty)$ such that $x(t) > 0$ for all $t \geq T$, and it will be said to be oscillatory if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $x(t_i) = 0$, $i = 1, 2, \cdots$, and $t_i \to \infty$ as $i \to \infty$.

LEMMA 2.1. *If, in addition to the conditions* (i) *and* (ii), *we assume that*

$$\text{(vi)} \qquad \int^{\infty} a(t)\,dt = \infty \left(\int^{\infty} b(t)\,dt = \infty\right),$$

then the inequality (1.4) ((1.3)) *cannot possess any solution which is ultimately positive* (*negative*).

Proof. Suppose to the contrary that there exists a solution $x(t)$ of (1.4) that is ultimately positive. Then for some t_0, $x(t) > 0$ for all $t \geq t_0$. We may re-write (1.4) as

$$\frac{x''}{\varphi_2(x)g(x')} + ka(t) \leq 0.$$

We now integrate from t_0 to t and obtain

$$\text{(2.1)} \qquad \frac{H(x'(t))}{\varphi_2(x(t))} - \frac{H(x'(t_0))}{\varphi_2(x(t_0))} + \int_{t_0}^{t} \frac{H(x'(s))\frac{d\varphi_2}{dx}x'(s)\,ds}{\varphi_2^2(x(s))} + k\int_{t_0}^{t} a(s)\,ds \leq 0\,.$$

where

$$H(x) = \int_0^x \frac{1}{g(v)}\,dv\,.$$

Since

$$xH(x) \geq 0\,, \quad \frac{d\varphi_2}{dx} \geq 0 \qquad \text{and} \qquad \int_{t_0}^{\infty} a(s)\,ds = \infty\,,$$

it follows from (2.1) that $\frac{H(x'(t))}{\varphi_2(x(t))} \to -\infty$ as $t \to \infty$, and hence $x'(t) < 0$ for large t. If $x'(t) \to -L$ as $t \to \infty$ $(0 < L < \infty)$, then it is obvious that $x(t)$ will be negative for large t, which contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. Since $x(t)$ and $x'(t)$ are of different signs for large t, $x(t)$ must then approach monotonically to a finite limit and $x'(t)$ cannot be bounded away from zero. The rest of the proof is similar to the one given by Wong [4] and is being given here for the sake of completeness.

Let $t^* \geq t_0$ be such that $x'(t) < 0$ for $t \geq t^*$.

Let N be the least positive integer such that $x'(t^*) < -\frac{1}{N}$.

We denote by t_N, the first $t \geq t^*$ for which $x'(t_N) = -\frac{1}{N}$ and define inductively t_{i+1} as the first $t \geq t_i$ for which $x'(t_{i+1}) = -\frac{1}{i+1}$, $i \geq N$.

Integrating

$$\frac{x''(t)}{g(x'(t))} \leq -ka(t)\,\varphi_2(x(t))$$

from t to t_i $(t_{i-1} \leq t < t_i)$ we obtain

$$H(x'(t_i)) - H(x'(t)) \leq -k\int_t^{t_i} a(s)\,\varphi_2(x(s))\,ds \tag{2.2}$$

$$= -k\varphi_2(x(t))\int_t^{t_i} a(s)\,ds - k\int_t^{t_i} \frac{d\varphi_2}{dx}\,x'(s)\left(\int_s^{t_i} a(\tau)\,d\tau\right)ds\,.$$

Let

$$y_i(t) = k\varphi_2(x(t))\int_t^{t_i} a(s)\,ds \quad , \quad \psi_i(t) = H(x'(t)) - H(x'(t_i))\,,$$

$$\psi(t) = -\frac{d\varphi_2}{dx}\,x'(t)/\varphi_2(x(t))\,.$$

Then (2.2) can be written as

$$y_i(t) \leq \psi_i(t) + \int_t^{t_i} \psi(s)\,y_i(s)\,ds\,. \tag{2.3}$$

Note that

$$\psi_i(t) = \int_{x'(t_i)}^{x'(t)} \frac{1}{g(v)}\,dv < 0$$

for i sufficiently large. From (2.3) we obtain, for $i \geq N$.

$$y_i(t) \leq \psi_i(t) + \int_t^{t_i} \psi_i(s)\,\psi(s)\exp\left(\int_t^{s} \psi(\tau)\,d\tau\right)ds.$$

Since $\psi_i < 0$ and $\psi(t) > 0$ for each $i \geq N$, $t_{i-1} \leq t_i < t_i, y_i(t_{i-1}) < 0$ for i sufficiently large, which contradicts the fact that

$$\varphi_2(x(t_{i-1})) > 0 \quad \text{and} \quad \int^{\infty} a(s)\, ds = \infty .$$

This completes the proof of the first part of the lemma. The second part can be proved by using a similar argument.

Remark. The result of the lemma remains true if $g(v)$ be replaced by $g(\gamma v)$ for any $\gamma > 0$.

LEMMA 2.2. *Let the condition* (vi) *be replaced by*

$$\text{(vii)} \quad \int^{\infty} ta(t)\, dt = \infty \quad , \quad a(t) \geq 0 \left[\int^{\infty} tb(t)\, dt = \infty , b(t) \geq 0 \right]$$

for large t, then the conclusion of Lemma 2.1 holds provided

$$\text{(viii)} \quad \int_{\alpha}^{\infty} \frac{1}{\varphi_2(u)}\, du < \infty \left(\int_{-\alpha}^{-\infty} \frac{1}{\varphi_1(u)}\, du < \infty \right) \quad \textit{for every} \quad \alpha > 0.$$

Proof. On the contrary let there exist a solution $x(t)$ of (1.4) which is ultimately positive. From (1.4) it follows that $x' < 0$ for $t \geq t_0$ for some t_0, so that $x'(t)$ is a nonincreasing function of t.

If $x'(t) \to -L$ $(0 < L \leq \infty)$ then it follows that $x(t) \to -\infty$ as $t \to \infty$ which contradicts the fact that $x(t) > 0$ for $t \geq t_0$. Hence $x'(t) \geq 0$ for $t \geq t_0$. Rewriting (1.4) as $\dfrac{tx''}{\varphi_2(x) g(x')} + kta(t) \leq 0$, and integrating from t_0 to t, we obtain

$$\frac{tH(x'(t))}{\varphi_2(x(t))} - t_0 \frac{H(x'(t_0))}{\varphi_2(x(t_0))} - \int_{t_0}^{t} \frac{H(x'(s))}{\varphi_2(x(s))}\, ds +$$

$$+ \int_{t_0}^{t} \frac{H(x'(s))\, s \dfrac{d\varphi_2}{dx} x'(s)}{\varphi_2^2(x(s))}\, ds + k \int_{t_0}^{t} sa(s)\, ds \leq 0 .$$

Whence it follows that

$$\frac{tH(x'(t))}{\varphi_2(x(t))} - \int_{t_0}^{t} \frac{H(x'(s))}{\varphi_2(x(s))}\, ds \to -\infty \text{ as } t \to \infty .$$

Since $x'(t) \geq 0$, it follows, using (v), that

$$H(x'(t)) \leq \frac{1}{c} x'(t) ,$$

and hence

$$\int_{t_0}^{t} \frac{H(x'(s))}{\varphi_2(x(s))} ds \leq \frac{1}{c} \int_{x(t_0)}^{x(t)} \frac{1}{\varphi_2(x(s))} ds < \infty .$$

Thus $\frac{tH(x'(t))}{\varphi_2(x(t))} \to -$ as $t \to \infty$, which contradicts the facts that $H(x'(t))$ and $\varphi_2(x(t))$ are nonnegative for large t.

The contradiction proves the first part of the lemma. The second part can be proved by a similar argument.

LEMMA 2.3. *If $g(v) \equiv 1$, then the result of the Lemma 2.2 holds even if the condition that $a(t) \geq 0$ ($b(t) \geq 0$) for large t be removed.*

Proof. Suppose there is a solution $x(t)$ of (1.3) which is ultimately positive. Then proceeding as in the proof of the Lemma 2.2 we can conclude that $\frac{tx'(t)}{\varphi_2(x(t))} \to -\infty$ as $t \to \infty$. Hence given $K > 0$ there exists a $T \geq t_0$ such that

$$\frac{tx'(t)}{\varphi_2(x(t))} < -K \qquad \text{for} \quad t \geq T,$$

or

$$\frac{x'(t)}{\varphi_2(x(t))} < -\frac{K}{t} \qquad \text{for} \quad t \geq T.$$

Integrating the above inequality from T to t, we get

$$\int_{T}^{t} \frac{x'(s)}{\varphi_2(x(s))} ds < -K \int_{T}^{t} \frac{1}{s} ds ,$$

or

$$\int_{x(T)}^{x(t)} \frac{1}{\varphi_2(u)} du < -K \log \frac{t}{T} .$$

From which it follows that $\int_{x(T)}^{x(t)} \frac{1}{\varphi_2(u)} du \to -\infty$, as $t \to \infty$, providing a contraction to the fact that $0 < \int_{\alpha}^{\infty} \frac{1}{\varphi_2(u)} du < \infty$.

Remark. In literature the function $\tau(t)$ is always taken to be a nonnegative continuous function of $t \in [0, \infty)$. Since we are not discussing Cauchy problems we will relax this restriction on $\tau(t)$.

THEOREM 2.4. *Let there exist function φ_1 and φ_2 satisfying the conditions* (i) *and* (iii), *then*

(α) *any solution $x(t)$ or a constant multiple of $x(t)$ of* (1.2), *which is ultimately positive satisfies the inequality* (1.4) *with $g(v)$ replaced by $g(\gamma v)$ for some $\gamma > 0$,*

(β) *any solution* $x(t)$ *or a constant multiple of* $x(t)$ *of* (1.2), *which is ultimately negative satisfies the inequality* (1.3) *with* $g(v)$ *replaced by* $g(\gamma v)$, *if necessary, for some* $\gamma > 0$.

Proof. Let $x(t)$ be a solution of (1.2) which is ultimately positive. Then for some t_0, $x(t) > 0$ for all $t \geq t_0$.

It follows that $x'' < 0$ for $t \geq t_0$, and hence $x'(t) \geq 0$ for $t \geq t_0$. Since $\tau(t) \leq \tau$ for all t, $x(t - \tau(t)) \geq x(t - \tau)$, for $t \geq t_0$. If $\tau \leq 0$, then $x(t) \leq \leq x(t - \tau)$ for $t \geq t_0 + |\tau|$, and

$$
\begin{aligned}
0 &= x'' + g(x')\,G(t, x(t - \tau(t)), x'(t'(t - \tau(t)))) \geq \\
&\geq x'' + g(x')\,G(t, x(t - \tau), x'(t - \tau)) \geq \\
&\geq x'' + g(x')\,G(t, x(t), 0) \geq \\
&\geq x'' + g(x')\,a(t)\,\varphi_2(x(t)).
\end{aligned}
$$

Thus $x(t)$ satisfies (1.4) with $k = 1$.

If $\tau > 0$, then $x(t) \geq x(t - \tau)$ for $t \geq t_0 + \tau$. Applying mean value theorem to $x(t)$ on $[t - \tau, t]$ we obtain

$$
\begin{aligned}
x(t) - x(t - \tau) &\leq x'(t_1)\,\tau \quad \text{for} \quad t - \tau < t_1 \leq t, \\
&\leq x'(t_0)\,\tau,
\end{aligned}
$$

or

$$\frac{x(t)}{x(t - \tau)} \leq 1 + \frac{x'(t_0)}{x(t - \tau)}\,\tau \leq 1 + \frac{x'(t_0)}{x(t_0)}\,\tau = \beta > 0,$$

or

$$x(t) \leq \beta(t - \tau) \qquad \text{for all} \quad t \geq t_0 + \tau.$$

Thus

$$
\begin{aligned}
0 &= x'' + g(x')\,G(t, x(t - \tau(t)), x'(t - \tau(t))) \geq \\
&\geq x' + g(x')\,G\left(t, \frac{1}{\beta}x(t), 0\right) \geq \\
&\geq x'' + g(x'')\,\varphi_2\left(\frac{1}{\beta}x(t)\right)a(t).
\end{aligned}
$$

Put $y(t) = \frac{1}{\beta}x(t)$, we get

$$y'' + \frac{1}{\beta}g(\beta y(t))\,a(t)\,\varphi_2(y(t)) \leq 0,$$

which proves the first part of the theorem. The part two can be established by a similar argument.

The proof of the following theorem follows that of Theorem 2.4.

THEOREM 2.5. *Suppose there exists functions* φ_1 *and* φ_2 *satisfying the conditions* (i) *and* (ii).

Then

(α) *any ultimately positive solution* $x(t)$ *of* (1.1) *satisfies the inequality* (1.4) *with* $g(v)$, *if necessary, replaced by* $g(\gamma v)$ *for some* $\gamma > 0$,

(β) *any ultimately negative solution of* (1.1) *satisfies the inequality* (1.3) *with* $g(v)$, *if necessary, replaced by* $g(\gamma v)$ *for some* $\gamma > 0$.

Proof. (α) As in the proof of Theorem 2.4, if $\tau \leq 0$,

$$\begin{aligned} 0 &= x'' + g(x')\,F(t, x(t), x(t-\tau(t))) \geq \\ &\geq x'' + g(x')\,F(t, x(t), x(t-\tau)) \geq \\ &\geq x'' + g(x')\,F(t, x(t), x(t)) \geq \\ &\geq x'' + g(x')\,a(t)\,\varphi_2(x(t)) \qquad \text{for} \quad t \geq t_0 + |\tau|\,. \end{aligned}$$

Which implies that $x(t)$ satisfies (1.4) with $k = 1$.

If $\tau > 0$, then $x(t) \leq \beta x(t-\tau)$, for $t \geq t_0 + \tau$ with β as in the proof of Theorem 2.4. Then

$$\begin{aligned} 0 &= x'' + g(x')\,F(t, x(t), x(t-\tau(t))) \geq \\ &\geq x'' + g(x')\,F(t, x(t), x(t-\tau)) \geq \\ &\geq x'' + g(x')\,F(t, x(t-\tau), x(t-\tau)) \geq \\ &\geq x'' + g(x')\,F\left(t, \frac{1}{\beta}x(t), \frac{1}{\beta}x(t)\right) \geq \\ &\geq x'' + g(x')\,a(t)\,\varphi_2\left(\frac{1}{\beta}x(t)\right), \quad \text{for} \quad t \geq t_0 + \tau\,. \end{aligned}$$

Whence if follows that

$$y'' + \frac{1}{\beta}g(\beta y(t))\,\varphi_2(y(t)) \leq 0\,.$$

The part two of the theorem can be proved by a similar argument.

Remark. In Theorem 2.5 also as in Theorem 2.4 we will understand that either $x(t)$ or a constant multiple of $x(t)$ satisfies (1.4).

3. OSCILLATION THEOREMS

We are now ready for our main results.

THEOREM 2.1. *Under the hypotheses of Lemma 2.1 Theorem 2.4 the equation* (1.2) *is oscillatory.*

Proof. On the contrary let $x(t)$ be a nonoscillatory solution of (1.2), that is, there is a $T \geq 0$ such that $x(t)$ is of the same sign for $t \geq T$. Then it must either satisfy the inequality (1.3) or (1.4). In either case we arrive at a contradiction.

THEOREM 3.2. *Under the hypotheses of Lemma 2.1 and Theorem 2.5 the equation* (1.1) *is oscillatory.*

THEOREM 3.3. *Under the hypotheses of Lemma 2.2 and Theorem 2.4 the equation* (1.2) *is oscillatory.*

THEOREM 3.4. *Under the hypotheses of Lemma 2.2 and Theorem 2.5 the equation* (1.1) *is oscillatory.*

Remarks. 1) For $g(v) \equiv 1$, $F = a(t)\, x(t - \tau(t))$ and $a(t) \geq 0$.

Theorem 3.2 contains Theorem 1 of Bradley [1]. Furthermore we do not require τ to be nonnegative.

2) If $F = [x(t - \tau)]^{\gamma}$, where γ is the ratio of two nonnegative odd integers, then Theorem 3.4 reduces to Theorem 1 of Gollowitzer [6].

3) The results of the lemmas can be used to discuss properties of the solutions of differential equations of the type

$$x'' + ka(t) f(x) g(x') = 0, \tag{3.1}$$

and

$$x'' + ka(t) h(x, x') = 0. \tag{3.2}$$

and some of the existing results, for example of Waltman [7], can be deduced.

4) Burkowski [3] also discussed equation (1.1) under assumptions different from those of ours.

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