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An a priori bound for the Cauchy problem in Banach space

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Analisi matematica. — *An a priori bound for the Cauchy problem in Banach space.* Nota di ADA ARDITO e PAOLO RICCIARDI (*) presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si dà un nuovo metodo basato su funzioni di Liapunov per ottenere maggiorazioni a priori per equazioni differenziali in spazi di Banach.

1. INTRODUCTION

Let Ω be an open subset of the Banach space X and let $\partial\Omega$ denote its boundary and $\overline{\Omega}$ its closure. By $\|x\|$ we denote the norm of $x \in X$. Let $A : D_A \subset X \to X$ be a linear operator which is the generator of a strongly continuous semigroup e^{tA} and let $f : [0, T] \times \Omega \to X$ be a continuous (1) mapping $(t, x) \to f(t, x)$.

We consider the problems

$$(1.1) \qquad \begin{cases} u' = Au + f(t, u) \\ u(0) = u_0 \end{cases}$$

$$(1.2) \qquad u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(t, u(s))\, ds.$$

A function u of class C^1, such that $u(t) \in D_A$, $\forall t \geq 0$ and $t \to Au(t)$ is continuous, and satisfying (1.1) is said to be a classical solution of (1.1). A solution of (1.2) is said to be a mild solution of (1.1). It is well known (see T. Kato [7]) that if $u_0 \in D_A$ and f is of class C^1 then every mild solution is classical.

In addiction if f is locally Lipschitz one has local existence; to have global existence it is sufficient to have a priori estimates. Such a priori estimates are obtained if $-f(t, \cdot)$ is accretive for each t (see e.g., F. Browder [1], T. Kato [7]) or if there is some principle of conservation of energy (see e.g., I. E. Segal [12], T. Kato [7]). Another case in which a priori bounds can be established is when $f(t, \cdot) \in K_B(X)$ (2) (see M. Iannelli [5]). In this case one demonstrates that if u is a solution of (1.2) then $t \to \|u(t)\|$ is decreasing.

In this work we have replaced the norm with Liapunov function which decreases along the solutions and we establish a generalization of the results of M. Iannelli [6]. At the end we will give examples that generalize equations of T. Carleman.

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(1) It is sufficient to require only that $f(t, \cdot)$ be integrable.

(2) $\|x\| \leq \|x - \alpha f(t, x)\| \quad \forall \alpha > 0 \quad \forall t.$

2. A THEOREM OF GLOBAL EXISTENCE

DEFINITION 2.1. *One says that the problem* (1.1) *or* (1.2) *has an a priori bound if for each solution*

$$u: \ [0, \alpha[\to \Omega \quad \alpha \leq T \tag{2.1}$$

the set

$$\mathcal{O} = \{u(t) : t \in [0, \alpha[\} \tag{2.2}$$

is bounded and is bounded a positive distance away from $\partial\Omega$.

THEOREM 2.2. *If the problem* (1.1) *is such that*:

(i) *there exists an a priori bound*;

(ii) *there exists a local solution for every* $u_0 \in \Omega$;

(iii) $f(t, u)$ *is bounded* (*that is, maps bounded sets into bounded sets*)

then there exists a global solution of (1.1).

Proof. By Zorn's lemma and (ii) there exists a function

$$u: \ [0, \alpha[\to \Omega \tag{2.3}$$

which is a maximal solution of (1.1). From (i) we have that $\mathcal{O}$ defined in (2.2) is bounded and from (iii) there exists $M \in \mathbf{R}^+$ such that

$$\|f(t, x)\| \leq M \qquad \forall (t, x) \in [0, \alpha[\times \mathcal{O}. \tag{2.4}$$

By (2.4) results that exists the Lebesgue integral

$$\int_0^\alpha e^{(\alpha-s)A} f(s, u(s)) \, ds. \tag{2.5}$$

Then for each $t \in [0, \alpha[$ one has

$$\left\| e^{\alpha A} u_0 + \int_0^\alpha e^{(\alpha-s)A} f(s, u(s)) \, ds - e^{tA} u_0 - \right. \tag{2.6}$$

$$\left. - \int_0^t e^{(t-s)A} f(s, u(s)) \, ds \right\| \leq \| e^{\alpha A} u_0 - e^{tA} u_0 \| +$$

$$+ \left\| \int_t^\alpha e^{(\alpha-s)A} f(s, u(s)) \, ds \right\| + \left\| \int_0^t [e^{(\alpha-s)A} f(s, u(s)) - e^{(t-s)A} f(s, u(s))] \, ds \right\|.$$

Therefore

$$\lim_{t \to \alpha} u(t) = u(\alpha). \tag{2.7}$$

On the other hand by (i) $u(\alpha) \in \Omega$. If follows by the maximality of $u(t)$ that $\alpha = T$.

3. A PRIORI BOUNDS (CLASSICAL SOLUTIONS)

LEMMA 3.1. *Let* $I \subset \mathbf{R}$, $V(t, x)$ *a mapping of* $I \times \Omega \to R$ *and* $u(t)$ *a mapping of* $I \to \Omega$. *If*

(i) $V(t, x)$ *is locally Lipschitz in* $x \in \Omega$, *uniformly in* t *on each bounded interval* $J \subset I$;

(ii) $\frac{\partial}{\partial t} V(t, x)$ *exists and is continuous in* (t, x);

(iii) $u(t)$ *is absolutely continuous on each bounded interval* $J \subset I$;

(iv) [(3)] $u(t)$ *is differentiable* a.e.

Then

(α) $V(t, u(t))$ *is differentiable* a.e.

(β) [(4)] $d/dt\, V(t, u(t)) = \frac{\partial}{\partial t} V(t, u(t)) + D_u^- V(t, u(t))\, u'(t)$ a.e.

Proof. Fix $\tau \in J$ and from the hypothesis it is possible to find an interval I_τ and two constants M_J and K_J such that for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that for each finite system of intervals $[\alpha_k, \beta_k]$ $(k = 1, \cdots, n)$ contained in I_τ such that $\sum_1^n{}_k (\beta_k - \alpha_k) < \delta_\varepsilon$ one has

$$\sum_1^n{}_k |V(\beta_k, u(\beta_k)) - V(\alpha_k, u(\alpha_k))| < \varepsilon. \tag{3.1}$$

In fact

$$\begin{aligned} \sum_1^n{}_k |V(\beta_k, u(\beta_k)) - V(\alpha_k, u(\alpha_k))| &\le \sum_1^n{}_k K_J \|u(\beta_k) - u(\alpha_k)\| + \\ + \left|\frac{\partial}{\partial t} V(\xi_k, u(\alpha_k))\right| |\beta_k - \alpha_k| &\le \sum_1^n{}_k K_J \|u(\beta_k) - u(\alpha_k)\| + M_J |\beta_k - \alpha_k| \text{ (5)} \\ &\qquad \alpha_k < \xi_k < \beta_k . \end{aligned} \tag{3.2}$$

It will suffice therefore to choose $\delta_\varepsilon \le \min\left(\delta_{\varepsilon'}, \frac{\varepsilon - K_J \varepsilon}{M_J}\right)$ where $\delta_{\varepsilon'}$ is relative to the absolute continuity of $u(t)$ in J. Therefore (α) is established.

With regard to (β) for the points where $V(t, u(t))$ is differentiable one has

$$\begin{aligned} d/dt\, V(t, u(t)) &= \lim_{h\to 0} 1/h\,[V(t+h, u(t+h)) - V(t, u(t))] = \\ &= \lim_{h\to 0} 1/h\,[V(t, u(t) + hu'(t)) - V(t, u(t))] + \\ &+ \lim_{h\to 0} 1/h\,[V(t, u(t+h) - V(t, u(t) + hu'(t))] + \\ &+ \lim_{h\to 0} 1/h\,[V(t+h, u(t+h)) - V(t, u(t+h))] . \end{aligned} \tag{3.3}$$

(3) If X is reflexive then (iii) $\Rightarrow$ (iv).

(4) $D_u^- V(t, u(t))\, u'(t) = \overline{\lim_{h\to 0}}\, 1/h\,[V(t, u(t) + hu'(t)) - V(t, u))]$.

(5) Since $\alpha_k, \beta_k \in \bar{J} \subset I$, one has by (ii) that there exists a constant M_J such that $\left|\frac{\partial}{\partial t} V(\xi_k, u(\alpha_k))\right| \le M_J$.

It follows from (ii)

$$(3.4) \qquad \lim_{h\to 0} 1/h\,[V(t+h\,,u(t+h)) - V(t\,,u(t+h))] = \lim_{h\to 0} 1/h\,\frac{\partial}{\partial t}V(\xi\,,u(t+h))\,h = \frac{\partial}{\partial t}V(t\,,u(t)) \qquad t+h<\xi<t$$

and from (i)

$$(3.5) \qquad \lim_{h\to 0} |\,1/h\,[V(t\,,u(t+h)) - V(t\,,u(t)-hu'(t))]\,| \leq \lim_{h\to 0} 1/|h|\,K_J\,\|u(t+h)-u(t)-hu'(t)\| = 0.$$

Therefore we have (β).

THEOREM 3.2. *If there exists a mapping* $V(t\,,x)$, $V:[0\,,T]\times\Omega\to\mathbf{R}$, *satisfying the hypotheses* (i) *and* (ii) *of Lemma 3.1. such that*

(iii) $\frac{\partial}{\partial t}V(t\,,x) + D_x^- V(t\,,x) f(t\,,x) \leq 0$

(iv) [6] u_0 *is such that the set*

$$(3.6) \qquad K_{u_0} = \bigcup_{t\in[0,T]} \{x/V(t\,,x)\leq V(0\,,u_0)\}$$

is bounded and bounded a positive distance away from $\partial\Omega$ *then there exists an a priori bound for the classical solutions of* (1.1).

Proof. Let $u(t)\,,u:[0\,,\alpha[\,\to\Omega$, be a classical solution of (1.1). Then $u(t)$ satisfies the hypothesis (iii) and (iv) of Lemma 3.1. Therefore from (iii) one obtains

$$(3.7) \qquad d/dt\,V(t\,,u(t))\leq 0 \quad \text{a.e.}$$

It follows that for each $t\in[0\,,\alpha[$

$$(3.8) \qquad V(t\,,u(t))\leq V(0\,,u_0).$$

Therefore $\mathcal{O}$ defined in 2 (2.2), is contained in K_{u_0}, and the conclusion follows.

LEMMA 3.3. [7] *Let* $u_0\in D_A\cap\Omega$. *If*

(i) $f(t,u)$ *has partial derivatives with respect to t and u and these partial derivatives are continuous in* $(t\,,u)$.

If the (1.1) *has a mild solution then this solution is classical.*
If in addition

(ii) $\frac{\partial}{\partial t}f(t\,,u)$ *is bounded.*

then there exists a local solution.

(6) If $\Omega = X$ then this condition can be replaced by the following.

(iv)′ $\lim_{\|x\|\to+\infty} V(t\,,x) > V(0\,,u)$ uniformly in t in $[0\,,T]$.

(7) See T. Kato [7].

COROLLARY 3.4. *Under the hypothesis of Lemma 3.3. and Theorem 3.2. If*

(i) $f(t, u)$ *is bounded*

there is a global classical solution of (1.1).

Proof. The results is obvious from Theorem 2.2. observing that by Lemma 3.3. $u(\alpha) \in D_A \cap \Omega$.

4. A PRIORI BOUNDS (MILD SOLUTIONS)

THEOREM 4.1. *Suppose there exists a sequence* $\{f_n\}$, $f_n : [0, T] \times \Omega \to X$, *of mappings verifying the hypotheses of Lemma 3.3., of Corollary 3.4. and such that.*

(i) $\{f_n\}$ *is equi–Lipschitz continuous on bounded subsets of* Ω *uniformly in* $t \in [0, T]$;

(ii) $\lim_n f_n(t, x) = f(t, x)$ *for each fixed* (t, x).

Suppose in addition there exists a mapping $V(t, x)$, $V : [0, T] \times \Omega \to \mathbf{R}$ *satisfying the hypothesis* (i) *and* (ii) *of Lemma 3.1. and*

(iii) $\forall n$, $\frac{\partial}{\partial t} V(t, x) + D_x^- V(t, x)(Ax + f_n(t, x)) \leq 0$

$(iv)_1$ $K_{u_0} = \bigcup_{t \in [0,T]} \{x / V(t, x) \leq V(0, u_0)\}$ *is bounded and bounded a positive distance away from* $\partial\Omega$

$(iv)_2$ $D_A \cap K_{u_0}$ *is dense in* K_{u_0}.

Then there is an a priori bound for (1.2).

Before proving Theorem 4.1. we first state the following Lemma whose proof is obvious.

LEMMA 4.2. *If* $(iv)_1$ *and* $(iv)_2$ *hold then for each* $u_{0_n} \in D_A \cap K_{u_0}$ *the set* $K_{u_{0_n}} = \bigcup_{t \in [0,T]} \{x / V(t, x) \geq V(0, u_{0_n})$ *is contained in* K_{u_0}.

Proof of 4.1. Since by $(iv)_2$, $D_A \cap K_{u_0}$ is dense in K_{u_0}, there exists a sequence $\{u_{0_n}\} \subset D_A \cap K_{u_0}$ that converges to u_0. We consider the problem

$$u_n(t) = u_{0_n} + \int_0^t e^{(t-s)A} f_n(s, u_n(s))\, ds.$$

By Lemma 4.2. and by (iii) above it follows that there is a global classical solution for each n.

Let $u(t)$, $u : [0, \alpha[\to \Omega$ be a local solution of (1.2). Then $\forall t \in [0, \alpha[$

$$\|u_n(t) - u(t)\| \leq \|e^{tA} u_{0_n} - e^{tA} u_0\| + \tag{4.1}$$

$$+ \left\| \int_0^t e^{(t-s)A} f_n(s, u_n(s)) - e^{(t-s)A} f_n(s, u(s))\, ds \right\| +$$

$$+ \left\| \int_0^t [e^{(t-s)A} f_n(s, u(s)) - e^{(t-s)A} f(s, u(s))]\, ds \right\|.$$

By the continuity of e^{tA} for each fixed t we have

$$\lim_{n\to\infty} \| e^{tA} u_{0_n} - e^{tA} u_0 \| = 0 \tag{4.2}$$

and by the boundedness of f_n and the Theorem of dominated convergence we have

$$\lim_{n\to\infty} \left\| \int_0^t [e^{(t-s)A} f_n(s, u(s)) - e^{(t-s)A} f(s, u(s))] \, ds \right\| = 0 . \tag{4.3}$$

Therefore it follows that

$$\| u_n(t) - u(t) \| \le \varepsilon_n + \left\| \int_0^t e^{(t-s)A} [f_n(s, u(s)) - f(s, u(s))] \, ds \right\| . \tag{4.4}$$

By (i) and the Theorem of Hille–Yosida there exists a constant c such that

$$\| u_n(t) - u(t) \| \le \varepsilon_n + c \int_0^t \| u_n(s) - u(s) \| \, ds. \tag{4.5}$$

By Gronwalls' Lemma

$$\| u_n(t) - u(t) \| \le \varepsilon_n e^{cT}. \tag{4.6}$$

Therefore $\lim_n u_n(t) = u(t)$ $\forall t \in [0, \alpha[$. By (iii)

$$V(t, u_n(t)) \le V(0, u_{0_n}) \tag{4.7}$$

and one has, passing to the limit

$$V(t, u(t)) \le V(0, u_0) \qquad \forall t \in [0, \alpha[\tag{4.8}$$

from which the conclusion follows by $(iv)_1$.

5. Applications

Example 5.1. We consider the second order equation

$$\begin{cases} \ddot{u} + f(u) = 0 \\ \dot{u}(0) = u_1 \quad u(0) = u_0 . \end{cases} \tag{5.1}$$

If there exists a mapping $F(x)$, $F : X \to \mathbf{R}$ such that

(i) $F(x)$ is locally Lipschitz;

(ii) $\{x_n\} \subset X$ is such that $\lim_{n\to\infty} \|x_n\| = +\infty$ then $\lim_n F(x_n) \ge 0$

(iii) (8) for each $y' \in \partial \| y \|$

$$\langle y , F'(x) \rangle = \| y \| \langle f(x) , y' \rangle . \tag{5.2}$$

Then there exists a mapping $V(x , y)$, $V : X \oplus X \to \mathbf{R}$ that verifies the hypothesis of Theorem 3.2.

Proof. We observe that for each $x' \in \partial \| \dot{u}(t) \|$ one has

$$\langle \ddot{u}(t) , x' \rangle + \langle f(u) , x' \rangle = 0. \tag{5.3}$$

By (iii) it follows that

$$\| \dot{u}(t) \| \langle \ddot{u}(t) , x' \rangle + \langle \dot{u}(t) , F'(u(t)) \rangle = 0. \tag{5,4}$$

By a result of T. Kato and by the chain rule one has

$$\langle \ddot{u}(t) , x' \rangle = d/dt \| \dot{u}(t) \| \quad \text{a.e.} ; \qquad \langle \dot{u}(t) , F'(u(t)) \rangle = d/dt\, F(u(t)). \tag{5.5}$$

Then

$$d/dt\, (1/2 \| \dot{u}(t) \|^2 + F(u(t))) = 0. \tag{5.6}$$

One may then choose

$$V(x , y) = 1/2 \| y \|^2 + F(x). \tag{5.7}$$

Example 5.2. We consider the system

$$\begin{cases} u = - \dfrac{\partial}{\partial x} u + \tau_1(u , v) & \\ v = - \dfrac{\partial}{\partial y} u + \tau_2(u , v) & t \in [0 , T] \\ u(0) = u_0(x , y) \quad v(0) = v_0(x , y). & \end{cases} \tag{5.8}$$

Let $X = C(0 , L) \oplus C[0 , L]$, $L \in \mathbf{R}^+$, $(u , v) \in X$, $\| (u , v) \| = \| u \| + \| v \|$, $\| u \| = \sup_{(x,y) \in [0,L]^2} | u(x , y) |$. Let A be the operator given by

$$A : D_A \subset X \to X \quad , \quad A(u , v) = \left(- \frac{\partial}{\partial x} u , - \frac{\partial}{\partial y} v \right)$$

$$D_A = \left\{ (u , v) \in X , \ \left(- \frac{\partial}{\partial x} u , - \frac{\partial}{\partial y} v \right) \in X \ \text{and} \ u(0 , y) = v(x , 0) = 0 \right\} .$$

Let $f : X \to X$ be given by $f(u , v) = (\tau_1(u , v) , \tau_2(u , v))$. Then (5.8) may be written, setting $w = (u , v)$, as

$$\begin{cases} \dot{w} = Aw + f(w) \\ w(0) = w_0 . \end{cases} \tag{5.9}$$

(8) If $X = H$ is a Hilbert space, (iii) implies that $F(x)$ is an anti-derivative of $f(x)$. In fact identifying H with H' in the usual way with the Theorem of Riesz, it follows that $F'(x)$ is a mapping of H to H such that $F'(u(t)) = f(u(t))$. In the particular case that $X = \mathbf{R}$, set $x = (x_1 , \cdots , x_n)$ and $f(x) = (f_1(x) \cdots , f_n(x))$ and by (iii) we may write

$$\sum_1^n {}_i \frac{\partial F(x)}{\partial x_i} y_i = \sum_1^n {}_i f_i(x) y_i .$$

If f verifies the hypothesis of Lemma 3.3. and if $w_0 \in D_A$, then (5.8) admits a classical local solution. If in addition (i) $\exists p, q \geq 1$, $\tau' \in \partial \| u \|$, $\xi' \in \partial \| v \|$ such that

$$p \| u \|^{p-1} \langle \tau_1(u, v), \tau' \rangle + q \| v \|^{q-1} \langle \tau_2(u, v), \xi' \rangle \leq 0$$

then there exists a mapping $V: X \to \mathbf{R}$ which assures the existence of an a priori bound for each $w_0 \in D_A$.

In fact one may choose

$$V(u, v) = \| u \|^p + \| v \|^q. \tag{5.10}$$

It follows that

(a) $$\lim_{\| u \|, \| v \| \to \infty} V(u, v) = +\infty$$

(b) $$D_w^- V(w) Aw = p \| u \|^{p-1} D_s \| u \| \left(-\frac{\partial}{\partial x} u\right) + q \| v \|^{q-1} D_s \| v \| \left(-\frac{\partial}{\partial y} v\right) =$$
$$= p \| u \|^{p-1} \inf_{\tau' \in \partial \| u \|} \left\langle -\frac{\partial}{\partial x} u, \tau' \right\rangle + q \| v \|^{q-1} \inf_{\xi' \in \partial \| v \|} \left\langle -\frac{\partial}{\partial y} u, \xi' \right\rangle \leq 0$$ [9]

(c) $$D_w^- V(w) f(w) = p \| u \|^{p-1} D_s \| u \| (\tau_1(u, v)) + q \| v \|^{q-1} D_s \| v \| (\tau_2(u, v)) =$$
$$= p \| u \|^{p-1} \inf_{\tau' \in \partial \| u \|} \langle \tau_1(u, v), \tau' \rangle + q \| v \|^{q-1} \inf_{\xi' \in \partial \| v \|} \langle \tau_2(u, v), \xi' \rangle \leq 0.$$

We observe that if $f(w) = f \circ w$ where $f \in C(\mathbf{R}) \oplus C(\mathbf{R})$ then by (i) we may write

$$p | x |^{p-1} \operatorname{sgn} x \, \tau_1(x, \bar{y}) + q | y |^{q-1} \operatorname{sgn} y \, \tau_2(\bar{x}, y) \leq 0 \qquad x > \bar{x}, y > \bar{y}.$$

For the problem e.g.

$$\begin{cases} u = -\dfrac{\partial}{\partial x} u - \dfrac{1}{4} u^3 - uv^2 \\ v = -\dfrac{\partial}{\partial y} u + u^6 - v^3 \end{cases}$$

one may choose $V: X \to \mathbf{R}$, $(u, v) \to V(u, v)$

$$V(u, v) = \| u \|^4 + \| v \|.$$

(9) We recall that $\partial \| u \|$ is the closed convex hull of the set

$$\{\tau' \in X', \tau' = \delta_{(x,y)^+} - \delta_{(x,y)^-}; (x, y)^+ \in M_0^+(u), (x, y)^- \in M_0^-(u)\}$$

where

$$M_0^+ = \{(x, y) \in \mathbf{R}^2, \quad u(x, y) \geq 0 \quad u(x, y) = \| u \|\}.$$

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