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Matematica. — *A Mean Value Theorem in Generalized Bi-Axially Symmetric Potential Theory.* Nota di DENNIS W. QUINN e RICHARD J. WEINACHT, presentata (*) dal Corrisp. G. FICHERA.

RIASSUNTO. — È dato un teorema di media per soluzioni della equazione della teoria generalizzata del potenziale simmetrico biassiale. Vengono anche date le analoghe diseguaglianze del valor medio in relazione alle corrispondenti inequazioni differenziali.

1. INTRODUCTION

In this Note a new mean value theorem is given for solutions of the equation of Generalized Bi-axially Symmetric Potential Theory (GBSPT)

$$\text{(1.1)} \qquad L[U] \equiv \sum_{i=1}^{n} U_{x_i x_i} + \frac{p}{x_{n-1}} U_{x_{n-1}} + \frac{q}{x_n} U_{x_n} = 0 .$$

In contrast to previous results in GBSPT (Kapilevich [6, 7]; see also the remark in Section 4 of Hall, Quinn and Weinacht [4]), the point at which the mean value is attained lies on only one of the two singular hyperplanes $x_{n-1} = 0$ or $x_n = 0$, and not on their intersection; moreover, this point is not located at the center of enclosing spheres. In the latter respect our result resembles Fichera's [3] mean value theorem for functions harmonic in a torus which also motivated Weinacht [11]. The present mean value theorem and its corollary for the corresponding differential inequalities are useful [10] in establishing properties of solution of (1.1) and related equations.

For the real constant $p = 0$, equation (1.1) reduces to Weinstein's [12, 13] Generalized Axially Symmetric Potential Theory (GASPT); q is also a real constant and n is an integer, $n \geq 2$. For GASPT mean value theorems have been given by Weinstein [12], Huber [5], and Weinacht [11] (see also Kapilevich [7, 8]). More recently Leschen [9] obtained interesting related results including converse theorems and a new proof of Weinacht's mean value theorem. We have adapted some of the ideas of his proof to obtain the present result. In a paper just published Diaz and Leschen [1] extend Weinstein's mean value theorem for GASPT to the range $-1 < q < 0$.

2. PRELIMINARIES

The usual notations for vectors in Euclidean n-space will be used. Let Q denote the open quarter space

$$Q = \{x \in R_n : x_{n-1} > 0 \ , \ x_n > 0\} .$$

For $b > 0$, let x^0 be the vector in R_n with components $x^0 = (0, \cdots, 0, b)$.

(*) Nella seduta del 20 aprile 1974.

For fixed b and $k > 1$, denote by Γ a member of the one-parameter family of hemispheres in Q with center at $(0, \cdots, 0, kb)$ and radius $r = b(k^2 - 1)^{1/2}$

$$\Gamma = \left\{ x \in R_n : \sum_{i=1}^{n-1} x_i^2 + (x_n - kb)^2 = b^2(k^2 - 1), x_{n-1} > 0 \right\}. \tag{2.1}$$

By H denote the corresponding open hemi ball.

3. THE MEAN VALUE THEOREM

THEOREM. *Let* G_1 *be an open connected set in* R_n *with* x^0 *in its interior, let* I *be the interior (in the topology of* R_{n-1}*) of* $\bar{G}_1 \cap \{x : x_{n-1} = 0, x_n > 0\}$ *and let* $G \equiv G_1 \cap Q$. *Then for* $p > 0$ *and* $q > 0$, *any solution* φ *of* (1.1) *in* G *belonging to* $C^2(G) \cap C'(G \cup I)$ *satisfies the mean value relation*

$$\varphi(x^0) = \frac{\Gamma\left(\frac{1}{2}[n+p]\right)}{\pi^{(n-1)/2}\,\Gamma\left(\frac{1}{2}[p+1]\right)} \frac{r_*^{1-n-p}}{W_1(k_*)} \int_{\Gamma_*} x_{n-1}^p (b^{-1} x_n)^{\frac{1}{2}(q-n-p)} \varphi(x)\, dS \tag{3.1}$$

where Γ_* *is any hemisphere of the family* (2.1) *corresponding to the parameter* $k_* > 1$ *and radius* r_* *such that the corresponding closed hemi-ball* $\bar{H}_*$ *is contained in* $G \cup I$. *The function* W_1 *is a hypergeometric function*

$$W_1(k_*) \equiv F\left(\frac{1}{2}(n+p-q), \frac{1}{2}(n+p+q-2); \frac{1}{2}(n+p); \frac{1}{2}(1-k_*)\right).$$

Remark. The hypothesis that φ belongs to $C'(G \cup I)$ can be replaced by that of assuming φ belongs to $C^2(G) \cap C^0(G \cup I)$ *and* that φ has bounded second partial derivatives in $G \cup I$. This follows by adapting the proof of Lemma 1.2.2. of [9].

Proof. Let $w_1(z) \equiv F(a, b; c; z)$ and w_2 be linearly independent solutions of the hypergeometric equation which are real valued for real negative z (for non-integral $c = \frac{1}{2}(n+p) > 1$ the solution w_2 can be chosen as $(-z)^{1-c} F(a-c+1, b-c+1; 2-c; z)$ with corresponding expressions [2] for integral $\frac{1}{2}(n+p)$ where the logarithmic case may arise).

Following [9] put

$$k \equiv \left[\sum_{i=1}^{n} x_i^2 + b^2\right] (2bx_n)^{-1}$$

and set

$$\varphi^*(x) \equiv x_n^{(2-n-p-q)/2}\, W(k) \tag{3.2}$$

where

$$W(k) \equiv w_1\left(\frac{1}{2}[1-k_*]\right) w_2\left(\frac{1}{2}[1-k]\right) - w_2\left(\frac{1}{2}[1-k_*]\right) w_1\left(\frac{1}{2}[1-k]\right) \tag{3.3}$$

and

$$a = \frac{1}{2}(n + p - q) \quad , \quad b = \frac{1}{2}(n + p + q - 2) \quad , \quad c = \frac{1}{2}(n + p) .$$

Then φ^* is a (real valued) solution of (1.1) in H_* which vanishes on Γ_* and is singular at x^0.

For

$$0 < \varepsilon < \frac{1}{2}(k_* - 1) \quad \text{and} \quad 0 < \delta < \frac{1}{2} b [(1 + \varepsilon)^2 - 1]^{1/2}$$

consider the region Ω bounded by Γ_*, the hyperplane $x_{n-1} = \delta$ and the hemisphere Γ_ε of the family (2.1) corresponding to the parameter $k = 1 + \varepsilon$. Then Green's second identity (using the exterior unit normal γ on the boundary $\partial\Omega$) for L defined by (1.1)

$$\int_\Omega x_{n-1}^p x_n^q (\varphi^* L[\varphi] - \varphi L[\varphi^*]) \, dx = \int_{\partial\Omega} x_{n-1}^p x_n^q \left(\varphi^* \frac{\partial\varphi}{\partial\gamma} - \varphi \frac{\partial\varphi^*}{\partial\gamma} \right) dS$$

yields for φ^* defined in (3.2) and the function φ given in the theorem

$$\int_{\Gamma_\varepsilon} x_{n-1}^p x_n^q \left(\varphi^* \frac{\partial\varphi}{\partial\gamma} - \varphi \frac{\partial\varphi^*}{\partial\gamma} \right) dS = \int_{\Gamma_*} x_{n-1}^p x_n^q \varphi \frac{\partial\varphi^*}{\partial\gamma} \, dS \tag{3.4}$$

since the boundary integrals over $x_{n-1} = \delta$ tend to zero as δ tends to zero from above and since $\varphi^* \equiv 0$ on Γ_*.

The mean value relation (3.1) follows from (3.4) by letting ε tend to zero and simplifying in the following way. On any curve of the family (2.1)

$$\frac{\partial\varphi^*}{\partial\gamma} = \left[(k^2 - 1)^{1/2} W'(k) + \frac{1}{2}(2 - n - p - q) \gamma_n W(k) \right] x_n^{-(n+p+q)/2} \tag{3.5}$$

with W given by (3.3). Hence, introducing hyperspherical coordinates with pole at $(0, \cdots, 0, (1 + \varepsilon) b)$ and letting ε tend to zero from above, the left side of (3.4) tends to

$$2^{n+p-2} (2 - n - p) \, b^{(n+p+q-2)/2} W_1(k_*) \frac{\pi^{(n-1)/2} \Gamma\left(\frac{1}{2}[p + 1]\right)}{\Gamma\left(\frac{1}{2}[n + p]\right)} \varphi(x^0) \tag{3.6}$$

where the cases of integral and non-integral $\frac{1}{2}(n + p)$ must be considered. From (3.5) the right side of (3.4) becomes

$$W'(k_*) (k_*^2 - 1)^{1/2} \int_{\Gamma_*} x_{n-1}^p x_n^{(q-n-p)/2} \varphi(x) \, dS. \tag{3.7}$$

Equation (3.7) may be simplified by observing that $W'(k_*)$ is the Wronskian of w and w_2 at k_* multiplied by the factor $\left(-\frac{1}{2}\right)$ so that

$$W'(k_*) = 2^{n+p-2} (2 - n - p) (k_*^2 - 1)^{-(n+p)/2} . \tag{3.8}$$

Combining (3.6) and (3.7) yields by use of (3.8) the mean value relation (3.1).

Remark 1. Putting $n = 2$ and letting p tend to zero through positive values in (3.1) yields Weinacht's mean value theorem [11] for GASPT in the form given by Leschen [9] for φ even in x_{n-1}.

Remark 2. If the point x^0 is on the plane $x_n = 0$, i.e. $x^0 = (0, \cdots, 0, b, 0)$, then an analogous mean value theorem is obtained from equation (3.1) by interchange of p and q, x_{n-1} and x_n, etc.

COROLLARY (Mean value inequality). *If* φ *in the theorem satisfies* $L[\varphi] \leq 0$ *(respectively* $L[\varphi] \geq 0$*) rather than* $L[\varphi] = 0$, *then* (3.1) *is valid provided equality is replaced by* $\geq$ *(respectively* $\leq$*).*

Proof. Following the proof of the theorem it is easy to see that (3.4) holds with equality replaced by the appropriate inequality, provided φ^* is non-negative in H_*. Then, upon simplification of (3.4) as before, the assertion follows.

To show that φ^* is non-negative in H_* is equivalent to showing that W in (3.3) is positive on $(1, k_*)$. First notice that $W_1(k)$ is positive for $k > 1$ as follows from (3.1) for $\varphi = 1$ or by use of the hypergeometric integral for real negative $z = \frac{1}{2}(1 - k_*)$. Now, recalling $W(k_*) = 0$, it is easy to see that W does not have a zero on $(1, k_*)$; for, if so, then Sturm's separation theorem applied to W and W_1 would require W_1 to have a zero on $(1, k_*)$, contradicting the positivity of W_1. Hence, W is of one sign on $(1, k_*)$ and this sign is positive since $W(k_*) = 0$ and $W'(k_*) < 0$ from (3.8). This completes the proof of the corollary.

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