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Group—like singularities in geometric theories.

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Topologia algebrica. — *Group–like singularities in geometric theories.* Nota di SANDRO BUONCRISTIANO, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — In questa Nota si annunciano risultati le cui dimostrazioni verranno pubblicate altrove.

Dato un gruppo abeliano G, si descrive la struttura che occorre introdurre nei (*co*)–cicli di una teoria di (*co*)–bordismo Ω (—) per ottenere Ω (—) con coefficienti in G''. Per amore di semplicità ci si concentra sul PL (*co*)–bordismo, benché la costruzione sia valida in generale.

NOTATION

We shall be working in the PL category throughout and will usually omit the prefix PL. Thus, for instance, manifold will stand for PL manifold homeomorphism for PL homeomorphism, etc.

$$c\,(X) \equiv \text{cone over } X; \qquad \text{component} \equiv \text{connected component};$$

$|n|$ = absolute value of the integer n; Y^p denotes an object Y of dimension p. If M is a manifold, ∂M is the set of boundary points of M. If X and Y are topological spaces, $X - Y$ is the subspace consisting of all the points of X which are not in Y.

Let G be an abelian group, $\theta_G : 0 \rightarrow R \rightarrow F \rightarrow G \rightarrow 0$ a free presentation of G; R_0 and F_0 bases in R and F respectively.

DEFINITION 1. A θ_G–*manifold of dimension* n consists of a compact polyhedral pair M, SM together with the following structure:

(1) $M - SM$ and SM are oriented manifolds of dimensions n and $n - 1$ respectively.

CONVENTION. We regard the empty set as a manifold of any dimension. Therefore SM may be empty in M.

(2) (Trivialisation). If V is a component of SM there is given a homeomorphism of pairs $h : (N, V) \rightarrow (V \times c\,(T), V)$ where T is a finite set of points and N is a regular neighbourhood of V in M.

Each component of $N - V$ will be called a *sheet near* V.

(3) (Labelling). Each component of $M - SM$ is labelled by one element of F_0 and each component of SM is labelled by one element of R_0.

(*) Nella seduta del 9 marzo 1974.

Thus, if V is a component of SM, each of the sheets near V inherits a label and an orientation from M — SM.

(4) (Compatibility). Suppose $V \subset SM$ is labelled by $r \in R_0$ with $r = \Sigma n_g g, g \in F_0, n_g \in Z$. Then, for each g, there are, near V, exactly $|n_g|$ sheets labelled by $g \in F_0$. Moreover the orientation of a sheet lebelled by g induces the given orientation on V if and only if $n_g > 0$.

It is easily checked that the subpolyhedron ∂M of M given by $\partial M = \partial(M - SM) \cup \partial SM$ is a θ_G–manifold of dimension $n - 1$ with structure (i.e. orientation, labelling, trivialisation) derived from the structure of M by restriction. We call ∂M the *boundary* of the θ_G–manifold M.

If M and M′ are θ_G–manifolds of dimension n, a θ_G*–embedding* $j: M \to M'$ is a PL embedding which preserves the structure.

Examples

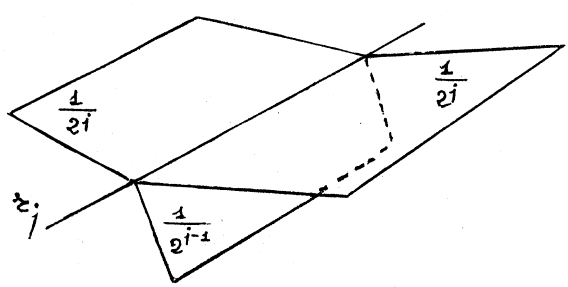
(1) $G = Z \qquad \theta_Z : 0 \to Z \xrightarrow{\text{id}} Z \to 0 \qquad F_0 = \{1 \in Z\}$.

There are no relations, hence no singularities. A θ_Z–manifold is an oriented manifold labelled by « 1 » $\in Z$.

(2) $G = Z_n \qquad \theta_{Z_n} : 0 \to Z \xrightarrow{n} Z \to Z_n \to 0 \qquad R_0 = 1 \qquad F_0 = 1$.

The singularities are now of the type $D^{n-1} \times c(T_n)$ where D^{n-1} is an $(n-1)$–dimensional disk and T_n is a set of n points. Thus θ_{Z_n}–manifolds are essentially the PL version of Sullivan's smooth Z_n–manifolds ([3]).

(3) $G = Z[1/2]$ (= additive group of those rational numbers whose denominators are powers of 2), $Z[1/2]$ is generated by $\left\{\frac{1}{2^i} : i = 0, 1, \cdots\right\}$ with a basic set of relations given by $\left\{r_j = \frac{1}{2^j} + \frac{1}{2^j} - \frac{1}{2^{j-1}} : j = 0, 1, \cdots\right\}$.



Germ of a $Z[1/2]$ singularity.

Therefore each component V of the singular set is labelled by a relation r_j and there are three sheets near V: two of them are labelled by $1/2^j$ the third is labelled by $1/2^{j-1}$ (see picture).

Now let X, A be a pair of topological spaces. *A singular* θ_G*–manifold* of dimension n is a pair (M, f) consisting of a θ_G–manifold M^n and a map $f: M, \partial M \to X, A$.

Using singular θ_G–manifolds and the notion of θ_G–embedding introduced before, one can define a bordism theory in the standard way (see for example [2]). We write $\Omega_*(X, A; \theta_G)$ for this generalized homology theory.

We have the following theorems.

THEOREM 1 (Universal-Coefficient sequence). *For each integer* $n \geq 0$ *and each pair* X, A *there exists a short exact sequence*:

$$0 \to \Omega_n(X, A) \otimes G \to \Omega_n(X, A; \theta_G) \to \operatorname{Tor}(\Omega_{n-1}(X, A), G) \to 0$$

which is natural in X, A □

THEOREM 2. *Up to equivalence of homology theories* $\Omega_*(X, A; \theta_G)$ *does not depend on any of the following choices*

(a) *the presentation* θ_G *of* G;

(b) *the bases* R_0 *and* F_0. □

Then, for each abelian group G, fix a presentation θ_G and define $\Omega_*(X, A; G)$, PL *oriented bordism with coefficients* G, by $\Omega_*(X, A; G) = \Omega_*(X, A; \theta_G)$.

THEOREM 3. *The homology theory* $\Omega_*(X, A; G)$ *is functorial on the category of abelian groups.* □

COROLLARY. *The Universal–coefficient sequence of Theorem 1 is natural on the category of abelian groups.* □

THEOREM 4 (The Bockstein sequence). *On the category of short exact sequences of abelian groups*

$$0 \to G' \xrightarrow{\psi} G \xrightarrow{\varphi} G'' \to 0$$

there is a natural connecting homomorphism

$$\beta : \Omega_n(X, A; G'') \to \Omega_{n-1}(X; A; G')$$

of degree -1 *and a natural long exact sequence*

$$\cdots \to \Omega_n(X, A; G') \xrightarrow{\psi_*} \Omega_n(X, A; G) \xrightarrow{\varphi_*} \Omega_n(X, A; G'') \to \cdots \quad \square$$

The proofs of the above theorems are entirely geometric in nature.

We now turn briefly to cobordism. In order to put coefficients G in cobordism we use the geometric description of cobordism given in [1]. Let K be an oriented cell complex, i.e. one in which each cell is oriented. Let G be an abelian group and θ_G a presentation of G.

DEFINITION 2. A (θ_G, q)–*mock bundle* ξ^q/K, with base K and total space E, is a projection $p : E \to |K|$ such that:

(a) for each $\sigma \in K$, $p^{-1}(\sigma)$ is the interior of a θ_G–manifold ξ_σ whose dimension equals $q + \dim \sigma$.

(b) for each $\sigma \in K$, $\xi_\sigma = \bigcup_{\tau \leqq \sigma} [\tau : \sigma]\, p^{-1}(\tau)$ where $[\tau : \sigma]$ is the incidence number of τ and σ.

(Here, if M is a θ_G–manifold, $-$M stands for the θ_G–manifold obtained from M by reversing all the orientations).

Using θ_G–mock bundles as in [1], a generalized cohomology theory $\Omega^*(|K|, G)$ is obtained and the cohomological versions of Theorems 1 to 4 can be proved.

REFERENCES

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[2] P. E. CONNER and E. E. FLOYD, *Differentiable periodic maps*, Springer–Verlag 1964.

[3] D. P. SULLIVAN, *Geometric Topology*, Part I: *Localization, Periodicity and Galois symmetry*, Lecture notes, «M.I.T.» (1970).