Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## Extending uniformly continuous functions

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **56** (1974), n.3, p. 318–321.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_3_318_0>

**Topologia.** — *Extending uniformly continuous functions* (*). Nota di ROBERT L. ELLIS, presentata (**) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si danno due Teoremi di estensione per funzioni uniformemente continue. Il primo teorema concerne le funzioni uniformemente continue da spazi uniformi non archimedei a spazi metrici completi. Come corollario si ha un risultato per le funzioni uniformemente continue analogo ad un teorema di Hausdorff per le funzioni continue tra spazi metrici. Il secondo teorema stabilisce che ogni sottoinsieme compatto di uno spazio topologico non archimedeo X è una retrazione uniforme di alcune strutture uniformi associate alla topologia di X.

## 1. INTRODUCTION

In three previous papers [1; 2; 3] the Author has given results concerning the extension of continuous functions and uniformly continuous functions on various types of zero-dimensional spaces. In the present paper two more extension theorems for uniformly continuous functions are given. The first involves uniformly continuous functions from non-Archimedean uniform spaces [6, Definition 4] into complete metric spaces. As a corollary we obtain an analogue for uniformly continuous functions of a theorem by Hausdorff [4, p. 41] concerning continuous functions between metric spaces. The second result involves non-Archimedean topological spaces, which were introduced by Monna [7, Definition 5] and characterized by Nyikos and Reichel [8, Theorem 17] in terms of uniform structures. We prove that every compact subset of a non-Archimedean topological space X is a uniform retract of X for any associated uniform structure on X (defined below).

## 2. UNIFORMLY CONTINUOUS FUNCTIONS ON NON-ARCHIMEDEAN UNIFORM SPACES

Let $\mathfrak{U}$ be a uniform structure on a set X,

An entourage $V \in \mathfrak{U}$ is *non-Archimedean* if $V^2 = V$.

We say $\mathfrak{U}$ is *non-Archimedean* if $\mathfrak{U}$ has a base of non-Archimedean entourages.

If $V \in \mathfrak{U}$, $x \in X$, and $A \subseteq X$, we write

$$V(x) = \{y \in X \mid (x, y) \in V\}$$

$$V(A) = \cup \{V(a) \mid a \in A\}.$$

(*) Research support by the University of Maryland and the German government through the auspices of the Alexander von Humboldt-Stiftung.

(**) Nella seduta del 9 marzo 1974.

An often used fact is that any point in $V(x)$ may be thought of as the " center " if V is non-Archimedean and symmetric; that is, if $y \in V(x)$, then $V(y) = V(x)$.

The following theorem generalizes a previous result [2, Theorem 5.4].

THEOREM 1. *Let* A *be a subset of a non-Archimedean uniform space* X *and let* Y *be a complete metric space. Every uniformly continuous function* $f: A \to Y$ *has a uniformly continuous extension* $g: X \to Y$.

*Proof.* Let d be a complete metric for Y. Since $f$ is uniformly continuous, there is a decreasing sequence $(V_n)_{n \geq 1}$ of symmetric non-Archimedean entourages, such that $V_1 = X \times X$ and for every $n \geq 2$, if $(x, y) \in V_n \cap (A \times A)$, then $d(f(x), f(y)) < 1/n$. For each $n \geq 1$ choose a complete set of representatives $\{ a_{i,n} \mid i \in I_n \}$ in A of the open partition $\{ V_n(x) \mid x \in A \}$ of $V_n(A)$ [2, Lemma 5.3].

For each $x \in X$ let

$$N(x) = \sup \{ n \mid x \in V_n(A) \}.$$

For $x \in X$ and $n$ an integer not greater than $N(x)$, choose $i(n, x) \in I_n$ such that $x \in V_n(a_{i(n,x),n})$. Let us abbreviate $a_{i(n,x).n}$ to $a(n, x)$. For each $n \geq 1$ define a function $g_n: X \to Y$ by

$$g_n(x) = \begin{cases} f(a(n, x)) & \text{if } N(x) = \infty \\ f(a(N(x), x) & \text{if } N(x) < \infty. \end{cases}$$

First we prove that each $x \in X$, $\{g_n(x) \mid n \geq 1\}$ is a Cauchy sequence in Y. This is obvious if $N(x) < \infty$.

If $N(x) = \infty$ and $m \geq n \geq 2$, then $(x, a(m, x)) \in V_m \subseteq V_n$ and $(x, a(n, x)) \in V_n$. Therefore $(a(m, x), a(n, x)) \in V_n^2 = V_n$ and so $d(g_m(x), g_n(x)) < 1/n$. Since d is a complete metric, we may define $g: X \to Y$ by

$$g(x) = \lim_{n \to \infty} g_n(x).$$

If $x \in A$ and $n > 2$, then $(x, a(n, x)) \in V_n$ and so $d(f(x), g_n(x)) < 1/n$. This implies that $g$ is an extension of $g$.

We will prove that $g$ is uniformly continuous by proving that if $n \geq 2$ and $(x, y) \in V_n$, then $d(g(x), g(y)) \leq 1/n$. Let $n \geq 2$ be given. Suppose first that $N(x)$ and $N(y)$ are both infinite. Then for $m \geq n$, $(x, a(m, x)) \in V_m \subseteq V_n$, $(x, y) \in V_n$, and $(y, a(m, y)) \in V_m \subseteq V_n$. Therefore, $(a(m, x), a(m, y)) \in V_n$, which implies that $d(g_m(x), g_m(y)) < 1/n$. Therefore, $d(g(x), g(y)) \leq 1/n$.

Next suppose that $N(x)$ and $N(y)$ are finite. Then $n$ cannot be strictly between $N(x)$ and $N(y)$, since if for example $N(x) < n < N(y)$, we would have $y \in V_n(A)$ and $(x, y) \in V_n$, which imply that $x \in V_n(A)$ and consequently that $N(x) \geq n$. If $n$ is not smaller that $N(x)$ and $N(y)$, then

$$(x, y) \in V_n \subseteq V_{N(x)} \cap V_{N(y)},$$

from which it follows easily that $N(x) = N(y)$ and hence that $g(x) = g(y)$. If $n$ is not larger than $N(x)$ and $N(y)$, then $(x, a(N(x), x)) \in V_{N(x)} \subseteq V_n$, $(x, y) \in V_n$ and $(y, a(N(y), y)) \in V_{N(y)} \subseteq V_n$. As before, it follows that $d(g(x), g(y)) \leq 1/n$. Finally, if $N(x)$ is finite and $N(y)$ is infinite, then $N(x) \geq n$ since $(x, y) \in V_n$ and $y \in V_n(A)$. For each $m \geq n$, we have $(y, a(m, y)) \in V_m \subseteq V_n$, $(x, y) \subseteq V_n$, and $(x, a(N(x), x)) \in V_{N(x)} \subseteq V_n$. Therefore, $(a(N(x), x), a(m, y)) \in V_n$. As before, $d(g(x), g(y)) \leq 1/n$.

A natural question is whether the result in Theorem 1 is true when Y is only assumed to be complete Hausdorff uniform space. We don't know the answer to that question. However, it is known [5, p. 188] that every Hausdorff uniform space can be uniformly embedded in the product of a family of complete metric spaces. Using this idea, we obtain as a corollary an extension theorem for uniformly continuous functions analogous to one obtained by Hausdorff [4, p. 41] for continuous functions on metric spaces.

COROLLARY 2. *Let* A *be a subset of a non-Archimedean uniform space* X *and let* Y *be a Hausdorff uniform space. If Y is uniformly embedded in the product* P *of a family of complete metric spaces* $\{Y_i \mid i \in I\}$, *then every uniformly continuous function* $f: A \to Y$ *has a uniformly continuous extension* $g: X \to P$.

*Proof.* For each $i \in I$ let $f_i : A \to Y$ be $f$ followed by the projection of P onto $Y_i$. Extend each $f_i$ to $g_i : X \to Y_i$ and let $g = \Pi \{g_i \mid i \in I\}$.

## 3. Uniformly continuous functions on non-Archimedean topological spaces

Monna [7, p. 178] introduced the concept of a non-Archimedean topological space, one having a base for its topology such that for any two sets U, V in the base either $U \cap V = \varphi$, $U \subseteq V$ or $V \subseteq U$. Such a base is called *non-Archimedean*. The prime examples of such spaces are the ultrametrizable spaces. Nyikos and Reichel presented other examples of non-Archimedean spaces [8, Example 1] and gave the following characterization in terms of uniform structures: a topological space X is non-Archimedean if and only if its topology is induced by a uniform structure having a base $\mathfrak{B}$ of non-Archimedean entourages such that $\{V(x) \mid V \in \mathfrak{B}, x \in X\}$ is a non-Archimedean base for X [8, Theorem 17]. We will call such a uniform structure an *associated uniform structure*.

THEOREM. 3. *Let* X *be a non-Archimedean topological space. Equip* X *with an associated uniform structure* $\mathfrak{U}$. *Let* A *be a compact subspace of* X. *Then there is a uniformly continuous retraction* $r: X \to A$.

*Proof.* Let $\mathfrak{B}$ be a base for $\mathfrak{U}$ consisting of symmetric non-Archimedean entourages such that $\mathfrak{N} = \{V(x) \mid V \in \mathfrak{B}, x \in X\}$ is a non-Archimedean base for X. We assume that $X \times X \in \mathfrak{B}$. For each $x \in X$ let

$$\mathfrak{B}_x = \{V \in \mathfrak{B} \mid V(x) \cap A \neq \varphi\}.$$

Since $\mathfrak{N}$ is a non-Archimedean base, it follows that $\{A \cap V(x) \mid V \in \mathfrak{B}_x\}$ is a collection of non-void closed subsets of A totally ordered by inclusion. As A is compact, the set.

$$A_x = \cap \{A \cap V(x) \mid V \in \mathfrak{B}_x\}$$

is not empty and $x \in A_x$ if $x \in A$. Choose $r(x) \in A_x$ subject to the restrictions that $r(x) = x$ if $x \in A$ and $r(x) = r(y)$ is $x$ and $y$ do not lie in A and $\mathfrak{B}_x = \mathfrak{B}_y$.

We will prove that $r$ is uniformly continuous by showing that for all $V \in \mathfrak{B}$, $(r(x), r(y)) \in V$ if $(x, y) \in V$. Notice that if $(x, y) \in V$, then $V \in \mathfrak{B}_x$ precisely when $V \in \mathfrak{B}_y$. Now if $(x, y) \in V$ and $V \in \mathfrak{B}_x$, then we have $r(x) \in A_x \subseteq \subseteq V(x)$, $r(y) \in V(y)$, and $(x, y) \in V$. As V is non-Archimedean, it follows that $(r(x), r(y)) \in V$. On the other hand, if $(x, y) \in V$ and $V \notin \mathfrak{B}_x$, if follows that $x$ and $y$ do not lie in A and $\mathfrak{B}_x = \mathfrak{B}_y$, for if $U \in \mathfrak{B}_x$, we have $U(x) \cap A \neq \varphi$ and $V(x) \cap A = \varphi$, which imply that $U(x)$ is not contained in $V(x)$. As $\mathfrak{N}$ is non-Archimedean, we conclude that $V(y) = V(x) \subseteq U(x)$. Thus, $y \in U(x)$ and so $U(x) = U(y)$. Therefore, $U \in B_y$ and so $\mathfrak{B}_x \subseteq \mathfrak{B}_y$. By symmetry, we obtain $\mathfrak{B}_x = \mathfrak{B}_y$. But then $r(x) = r(y)$.

COROLLARY 4. *Let* A *be a compact subset of a non-Archimedean topological space* X *equipt with an associated uniform structure. Let* Y *be a uniform space. Then every uniformly continuous function* $f : A \to Y$ *has a uniformly continuous extension* $g : X \to Y$.

COROLLARY. 5. *Let* A *be a compact subset of a non-Archimedean topological space* X. *Let* Y *be a topological space. Then every continuous function* $f : A \to Y$ *has a continuous extension* $g : X \to Y$.

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