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Translation 4-gonal configurations

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Geometrie finite. — *Translation 4-gonal configurations.* Nota di JOSEPH ADOLPHE THAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Introdotte certe configurazioni 4-gonali di traslazione, in relazione alla transitività del gruppo G delle loro traslazioni, si mostra la loro equivalenza coll'esistenza in G di taluni T-insiemi formati da sottogruppi, si definisce un campo F detto il loro nucleo e si stabilisce che G può venire considerato come il gruppo additivo di uno spazio vettoriale sopra un sottocampo F′ di F. In ogni caso $[G : F'] = 3n$, con n intero positivo, e la configurazione ammette una certa struttura canonica. Se $[G : F] = 3$, la configurazione vien detta desarguesiana ed essa può venire caratterizzata dalla proprietà di risultare isomorfa ad una configurazione 4-gonale di Tits.

1. INTRODUCTION AND REVIEW

A finite 4-gonal configuration of order s (≥ 1) [2] is an incidence structure $S = (P, B, I)$, with an incidence relation satisfying the following axioms:

(i) each point is incident with $s + 1$ lines and two distinct points are incident with at most one line;

(ii) each line is incident with $s + 1$ points and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there are a unique point x' and a unique line L' such that $x \mathrm{I} L' \mathrm{I} x' \mathrm{I} L$.

If $S = (P, B, I)$ is a 4-gonal configuration of order $s (\geq 1)$, then $|P| = |B| = s^3 + s^2 + s + 1$ [2].

Let $S = (P, B, I)$ be a 4-gonal configuration of order s (≥ 1) and let x and y be distinct points of S. The trace of x and y is defined to be the set $tr(x, y) = \{$ all $z \in P \parallel z$ is collinear with both x and $y\}$. The pair (x, y) of distinct points is said to be regular provided each point collinear with at least two points of $tr(x, y)$ is actually collinear with all points of $tr(x, y)$ (if the points x and y $(x \neq y)$ are collinear then evidently the pair (x, y) is regular). If (x, y) is a regular pair, the span of x and y is defined unambiguously as the set $sp(x, y) = tr(z, w)$ for any distinct points z, w in $tr(x, y)$ (for collinear points x and y $(x \neq y)$, we have $sp(x, y) = tr(x, y)$). When for a point x each pair (x, y), $x \neq y$, is regular, x is said to be regular. These definitions are easily dualized for lines. A point (resp. line) is called coregular provided each line (resp. point) incident with it is regular. And a point or line which is both regular and coregular is said to be biregular (then necessarily s is even [5]).

Let x be a regular point of a 4-gonal configuration S of order $s (> 1)$. Let Π_x be the incidence structure whose points are the points of S collinear with x and whose lines are the spans of the (necessarily regular) pairs of distinct points of Π_x. Then Π_x with the natural incidence relation is a projective plane of order s [6]. This theorem is easily dualized for a regular line L of S.

2. TRANSLATION 4-GONAL CONFIGURATIONS

2.1. TRANSLATIONS. Let $S^{(x_\infty)} = (P, B, I)$ be a 4-gonal configuration of order $s (> 1)$ which possesses a coregular point x_∞. The $s + 1$ regular lines $L_1, \cdots, L_{s+1}$ which are incident with x_∞ are called the ideal lines of $S^{(x_\infty)}$, and

(*) Nella seduta del 9 marzo 1974.

the s^2+s+1 points which are collinear with x_∞ are called the ideal points of $S^{(x_\infty)}$. We also introduce the following notations:

$$\mathcal{L}_\infty = \{L_1, L_2, \cdots, L_{s+1}\} \quad \text{and} \quad P_\infty = \{\text{all ideal points of } S^{(x_\infty)}\}.$$

A collineation θ of $S^{(x_\infty)}$ is called a translation if θ is the identity or if θ is a collineation satisfying $L_i^\theta = L_i \; \forall L_i \in \mathcal{L}_\infty$ (this implies that $x_\infty^\theta = x_\infty$) and $x^\theta \neq x \; \forall x \in P \setminus P_\infty$.

2.2. THEOREM. *A translation θ of $S^{(x_\infty)}$ induces a translation θ_i of the affine plane $\Pi_{L_i}^{\mathcal{L}_\infty}$ ($\mathcal{L}_\infty$ is the ideal line of this affine plane), $i \in \{1, 2, \cdots, s+1\}$.*

Proof. The points of the projective plane Π_{L_i} are the s^2+s+1 lines of $S^{(x_\infty)}$ which are concurrent with L_i. Lines of Π_{L_i} are (*a*) the line $\mathcal{L}_\infty$ (*b*) the sets $\mathcal{L}_j^i$ with elements the $s+1$ lines of $S^{(x_\infty)}$ incident with the point $x_{i,j} \neq x_\infty$, where $x_{i,j} \, \mathrm{I} L_i$ ($j = 1, \cdots, s$) (*c*) the sets with elements the $s+1$ lines of $S^{(x_\infty)}$ which are concurrent with L_i and a line $L \in B$ not concurrent with L_i (i.e. the sets $tr(L_i, L)$, $L \in B$ and L_i, L not concurrent).

The translation θ induces a permutation θ_i of the pointset of the projective plane Π_{L_i}. Moreover $L_j^{\theta_i} = L_j$, $j = 1, \cdots, s+1$. Now we show that θ_i is a collineation of the projective plane Π_{L_i}.

First of all we remark that $\mathcal{L}_\infty^{\theta_i} = \mathcal{L}_\infty$. Next we consider a line $\mathcal{L}_j^i$ of type (*b*). Then $(\mathcal{L}_j^i)^{\theta_i} = \mathcal{L}_{j'}^i$, where $x_{i,j}^\theta = x_{i,j'}$. Finally we consider a line $\mathcal{L}$ of type (*c*) of Π_{L_i}, where $\mathcal{L} = tr(L_i, L)$, $L \in B$ and L_i, L not concurrent. Then the elements $M_1, M_2, \cdots, M_s, L_j (i \neq j)$ of $\mathcal{L}$ are mapped by θ onto the lines $M_1^\theta, M_2^\theta, \cdots, M_s^\theta, L_j^\theta = L_j$ which are concurrent with $L_i^\theta = L_i$ and L^θ (remark that L^θ is not concurrent with L_i). Consequently $\mathcal{L}^{\theta_i} = \{M_1^\theta, M_2^\theta, \cdots, M_s^\theta, L_j\} = tr(L_i, L^\theta)$ is a line of type (*c*) of Π_{L_i}. We conclude that θ_i is a collineation of the projective plane Π_{L_i}. As $L_j^{\theta_i} = L_j$, $\forall L_j \in \mathcal{L}_\infty$, θ_i can be considered as a dilatation of the affine plane $\Pi_{L_i}^{\mathcal{L}_\infty}$. Next we prove that θ_i is a translation of the affine plane $\Pi_{L_i}^{\mathcal{L}_\infty}$.

For that purpose we suppose that θ is not the identity and that $L^\theta = L$ or $L^{\theta_i} = L$, where L is a point of Π_{L_i} and $L \notin \mathcal{L}_\infty$. We remark that $x_{i,j}^\theta = x_{i,j}$, with $L \mathrm{I} x_{i,j} \mathrm{I} L_i$. Next let y be a point of $P \setminus P_\infty$ which is not collinear with $x_{i,j}$. Call M and z the elements defined by $y \mathrm{I} M \mathrm{I} z \mathrm{I} L$ and M' and z' the elements defined by $y^\theta \mathrm{I} M' \mathrm{I} z' \mathrm{I} L$ (remark that $z \neq x_{i,j}$). Then evidently $M' = M^\theta$ and $z' = z^\theta$ (remark that $z^\theta \neq z$ and $M^\theta \neq M$). If M is concurrent with L_k ($i \neq k$), then also $M' = M^\theta$ is concurrent with L_k (we also remark that the points m and m', defined by $M \mathrm{I} m \mathrm{I} L_k$ and $M' \mathrm{I} m' \mathrm{I} L_k$ ($m' = m^\theta$), are distinct). Next let N and u be defined by $y \mathrm{I} N \mathrm{I} \, u \mathrm{I} L_i$ and N' and u' be defined by $y^\theta \mathrm{I} N' \mathrm{I} u' \mathrm{I} L_i$ (remark that $u \notin \{x_\infty, x_{i,j}\}$). Then evidently $N' = N^\theta$ and $u' = u^\theta$. As L_k is regular there exists a point v which is incident with N and M', and there also exists a point v' which is incident with N' and M. Let us suppose a moment that $u \neq u^\theta$ (then $v \neq y^\theta$ and $y \neq v'$) and consider the dilatation θ_l of the affine plane $\Pi_{L_l}^{\mathcal{L}_\infty}$, $l \neq i$. Let $x_{l,j} \mathrm{I} L_l$ with $x_{l,j} \neq x_\infty$, call a and A the elements defined by $x_{l,j} \mathrm{I} A \mathrm{I} \, a \mathrm{I} L$, and call $x_{l,j'}$ and A' the elements defined by $a^\theta \mathrm{I} \mathrm{A}' \mathrm{I} x_{l,j'}$, IL_l. Then evidently $x_{l,j'} = x^\theta_{l,j}$. Now from $a \neq a^\theta$ there follows that $x^\theta_{l,j} \neq x_{l,j}$. Consequently the only fixed points of the dilatation θ_l are the ideal points $\mathrm{L}_1, \mathrm{L}_2, \cdots, \mathrm{L}_{s+1}$ of the affine plane $\Pi^{L_\infty}_{\mathrm{L}_l}$. There results that the dilatation θ_l is a translation of the affine plane $\Pi^{L_\infty}_{\mathrm{L}_l}$. As A, $\mathrm{A}' = \mathrm{A}^\theta = \mathrm{A}^{\theta_l}$, L_i are collinear points of the projective plane Π_l, the ideal point L_i is the center of the translation θ_l of the affine plane $\Pi^{L_\infty}_{\mathrm{L}_l}$. Now we suppose that $l \notin \{i, k\}$. Let b and B be defined by $y\mathrm{IBI}\, b\mathrm{IL}_l$ and b' and B' by $y^\theta \mathrm{IB}' \mathrm{I} b' \mathrm{IL}_l$. Then $\mathrm{B}' = \mathrm{B}^\theta = \mathrm{B}^{\theta_l}$ and $b' = b^\theta$. From the preceding there follows that $b \neq b^\theta$ and that B, B', L_i are collinear points of the projective plane Π_{L_l}. There follows that there exists a point $c \in \mathrm{P}$ such that $\mathrm{NI}\, c\mathrm{IB}'$. So we obtain a triangle in $\mathrm{S}^{(x_\infty)}$ (with vertices c, v, y^θ and sides M^θ, B^θ, N), a contradiction. So we conclude that $u = u^\theta$. There follows immediately that $v = y^\theta$ and $v' = y$ and so $\mathrm{N} = \mathrm{N}^\theta = \mathrm{N}^{\theta_i}$. As $\mathrm{N} = \mathrm{N}^{\theta_i}$ and $\mathrm{L} = \mathrm{L}^{\theta_i}$ ($\mathrm{L} \notin L_\infty$ and $\mathrm{N} \notin L_\infty$), the dilatation θ_i of the affine plane $\Pi^{L_\infty}_{\mathrm{L}_i}$ is the identity.

From the preceding there follows that each translation θ of $\mathrm{S}^{(x_\infty)}$ induces a translation θ_i of the affine plane $\Pi^{L_\infty}_{\mathrm{L}_i}$.

2.3. THEOREM. *The set* G *of all translations of* $\mathrm{S}^{(x_\infty)}$ *is a group.*

Proof. If θ is a translation of $\mathrm{S}^{(x_\infty)}$ then evidently θ^{-1} is also a translation of $\mathrm{S}^{(x_\infty)}$.

Next let θ, θ' be two translations of $\mathrm{S}^{(x_\infty)}$. Then $\mathrm{L}_i^{\theta\theta'} = \mathrm{L}_i$, $\forall \mathrm{L}_i \in L_\infty$, and $\theta\theta'$ induces the translation $\theta_i\,\theta_i'$ of the affine plane $\Pi^{L_\infty}_{\mathrm{L}_i}$, $i \in \{1, 2, \cdots, \cdots, s+1\}$. Now let us suppose that $x^{\theta\theta'} = x$, $x \in \mathrm{P} \setminus \mathrm{P}_\infty$. Call x_i and M_i the elements defined by $x\mathrm{IM}_i \mathrm{I} x_i \mathrm{IL}_i$, $i = 1, 2, \cdots, s+1$. Then evidently $\mathrm{M}_i^{\theta\theta'} = \mathrm{M}_i^{\theta_i\theta_i'} = \mathrm{M}_i$, $i = 1, 2, \cdots, s+1$. Consequently the translation $\theta_i\,\theta_i'$ of $\Pi^{L_\infty}_{\mathrm{L}_i}$ is the identity, $i = 1, 2, \cdots, s+1$. If follows that $\mathrm{L}^{\theta\theta'} = \mathrm{L}$, $\forall \mathrm{L} \in \mathrm{B}$, and so $y^{\theta\theta'} = y$, $\forall y \in \mathrm{P}$. Hence the collineation $\theta\theta'$ of $\mathrm{S}^{(x_\infty)}$ is the identity. We conclude that for any two translations θ, θ' of $\mathrm{S}^{(x_\infty)}$, the product $\theta\theta'$ is also a translation of $\mathrm{S}^{(x_\infty)}$.

From the preceding there follows immediately that the set G of all translations of $\mathrm{S}^{(x_\infty)}$ is a group.

2.4. REMARK. If G_i is the group of all translations of the plane $\Pi^{L_\infty}_{\mathrm{L}_i}$, then $\Phi_i : \mathrm{G} \to \mathrm{G}_i$, $\theta \to \theta_i$ is a homomorphism of G into G_i ($i = 1, 2, \cdots \cdots, s+1$).

2.5. THEOREM. *If* x *and* y *are elements of* $\mathrm{P} \setminus \mathrm{P}_\infty$ *then there is at most one translation* θ *of* $\mathrm{S}^{(x_\infty)}$ *for which* $x^\theta = y$.

Proof. If $x^\theta = y$ and $x^{\theta'} = y$, then $x^{\theta\theta'^{-1}} = x$. Hence $\theta\theta'^{-1}$ is the identity, and so $\theta = \theta'$.

2.6. DEFINITION. If the group G of all translations of $\mathrm{S}^{(x_\infty)}$ is transitive on $\mathrm{P} \setminus \mathrm{P}_\infty$, then we say that $\mathrm{S}^{(x_\infty)}$ is a translation 4-gonal configuration (from 2.5. there follows that the translation group G of a translation 4-gonal configuration $\mathrm{S}^{(x_\infty)}$ is sharply transitive on $\mathrm{P} \setminus \mathrm{P}_\infty$).

2.7. THEOREM. *Let* $S^{(x_\infty)}$ *be a translation 4-gonal configuration. Then* (*a*) *the order* s *of* $S^{(x_\infty)}$ *is a prime power* p^h; (*b*) *the group* G *of all translations of* $S^{(x_\infty)}$ *is an elementary abelian group.*

Proof. If $S^{(x_\infty)}$ is a translation 4-gonal configuration then the group $G^{\Phi_i} = G_i'$ is transitive on the points of the affine plane $\Pi_{L_i'}^{L_\infty}$ ($i = 1, 2, \cdots, s+1$). Consequently $\Pi_{L_i'}^{L_\infty}$ is a translation plane and G_i' is the group G_i of all translations of $\Pi_{L_i'}^{L_\infty}$ ($i = 1, 2, \cdots, s+1$) [2]. Hence the order s of $\Pi_{L_i'}^{L_\infty}$ (i.e. the order of $S^{(x_\infty)}$) is a prime power p^h, and $G_i = G^{\Phi_i}$ is an elementary abelian group [2].

Next we consider the mapping $\Theta : G \to G_i \times G_j$, $\theta \to (\theta_i, \theta_j)$ ($i \neq j$). It is easy to show that Θ is a monomorphism of G into $G_i \times G_j$. As G_i and G_j are elementary abelian groups of order $s^2 = p^{2h}$, there follows immediately that G is an elementary abelian group (of order $s^3 = p^{3h}$).

3. THE 4-GONAL CONFIGURATIONS $G(T)$

3.1. *T-sets.* Let G be an abelian group of order $s^3 (s > 1)$. Suppose that $T = \{H_1, H_2, \cdots, H_{s+1}\}$ is a set of $s+1$ subgroups of order s of G, and that $H_i H_j H_k = G \ \forall i, j, k$ with i, j, k distinct. Such a set T is called a T–set of the abelian group G.

First of all we remark that $H_i \cap H_j = \{1\}$, $i \neq j$. The cosets of the subgroup H_i, $i = 1, 2, \cdots, s+1$, are denoted by $H_i = H_{i,1}, H_{i,2}, \cdots, H_{i,s^2}$ (so we obtain $s^3 + s^2$ cosets). Each coset contains s elements of G and through each element of G there pass $s+1$ cosets. From $H_i \cap H_j = \{1\}$, $i \neq j$, there follows immediately that two different cosets have at most one element in common. As $H_i H_j H_k = G$ ($\forall i, j, k$ with i, j, k distinct), there do not exist three cosets $H_{i,j}, H_{i',j'}, H_{i'',j''}$ with $H_{i,j} \cap H_{i',j'} = \{a''\}$, $H_{i',j'} \cap H_{i'',j''} = \{a\}$, $H_{i'',j''} \cap H_{i,j} = \{a'\}$ and a, a', a'' distinct elements of G. So the tactical configuration with as points the s^3 elements of G, with as lines the $s^3 + s^2$ cosets $H_{i,j}$, and with the natural incidence relation, does not possess triangles.

Now we consider the cosets of H_i, $i \in \{1, 2, \cdots, s+1\}$, having an element in common with the set $H_1 \cup H_2 \cup \cdots \cup H_{s+1}$. From the preceding there follows that in this way we obtain $s^2 - s + 1$ cosets (one of these cosets is the group H_i). So there remain $s - 1$ cosets $H_{i,j_1}, H_{i,j_2}, \cdots, H_{i,j_{s-1}}$ of H_i. The set $H_i \cup H_{i,j_1} \cup \cdots \cup H_{i,j_{s-1}}$ is denoted by H_i^* ($i = 1, 2, \cdots, s+1$).

3.2. THEOREM. *The set* H_i^* *is a subgroup* (*of order* s^2) *of* G ($i = 1, 2, \cdots, s+1$).

Proof. Consider the natural homomorphism $\sigma_i : G \to G/H_i$, $a \to a'$. We introduce the following notations: $H_i^{\sigma_i} = \{1'\}$, $H_j^{\sigma_i} = H_j'$ ($j \neq i$), $H_{i,j_1}^{\sigma_i} = \{h_1'\}$, $H_{i,j_2}^{\sigma_i} = \{h_2'\}, \cdots, H_{i,j_{s-1}}^{\sigma_i} = \{h_{s-1}'\}$, $H_i' = \{1', h_1', h_2', \cdots, h_{s-1}'\}$. We remark that $H_k' \cap H_l' = \{1'\}$ ($k \neq l$), that $H_1' \cup H_2' \cup \cdots \cup H_{s+1}' = G/H_i$, that H'_j $(j \neq i)$ is a subgroup of order s of the group G/H_i (of order s^2), and that $H'_k H'_l = G/H_i$ $(k \neq l, k \neq i, l \neq i)$. Now we prove that H'_l is a subgroup (of order s) of G/H_i.

Let h'_k be an element of H'_i. If $h'^{-1}_k \notin H'_i$, then $h'^{-1}_k \in H'_l$ for some l $(l \neq i)$ and so $h'_k \in H'_l$, a contradiction. Now we consider two elements h'_k, h'_l of H'_i. Suppose a moment that $h'_k h'_l \notin H'_i$. Then $h'_k, h'_l \in H'_j$ for some $j \neq i$. We have $h'_k = u'_n v'_n$, with $u'_n \in H'_j \setminus \{1'\}$, $v'_n \in H'_n \setminus \{1'\}$, $n \in \{1, 2, \cdots, s+1\} \setminus \{i, j\}$. If $n \neq n'$ $(n' \in \{1, 2, \cdots, s+1\} \setminus \{i, j\})$, then evidently $u'_n \neq u'_{n'}$. As $|\{1, 2, \cdots, s+1\} \setminus \{i, j\}| = s - 1$ there exists a $m \in \{1, 2, \cdots, s+1\} \setminus \{i, j\}$ such that $u'_m = h'_k h'_l$. There results that $h'_k = h'_k h'_l v'_m$, with $v'_m \in H'_m \setminus \{1'\}$. Hence $h'_l = v'^{-1}_m \in H'_m \setminus \{1'\}$, a contradiction. Consequently $h'_k h'_l \in H'_i$, and so H'_i is a subgroup of G/H_i.

We conclude that $H_i'^{\sigma_i^{-1}} = H_i^*$ is a subgroup (of order s^2) of G.

3.3. REMARKS. 1) $H_i^* \cap H_j = \{1\}$ and $H_i^* H_j = G$ $(i \neq j)$. 2) The subgroups $H'_i, H_j^{\sigma_i}$ $(j \neq i)$ constitute a congruence partition of the group G/H_i (of order s^2). Consequently with each subgroup H_i of G there corresponds a translation plane of order s [2].

3.4. THEOREM. *If the abelian group* G *of order* $s^3 (s > 1)$ *has a T-set, then s is a prime power* p^h *and* G *is an elementary abelian group.*

Proof. From 3.3. there follows immediately that G/H_i is an elementary abelian group, and so s is a prime power p^h.

Now we consider the mapping $\Theta : G \to G/H_i \times G/H_j$, $a \to (a^{\sigma_i}, a^{\sigma_j})$ $(i \neq j)$. Evidently Θ is a monomorphism of G into $G/H_i \times G/H_j$. As G/H_i and G/H_j are elementary abelian groups of order $s^2 = p^{2h}$, there follows immediately that G is an elementary abelian group of order p^{3h}.

3.5. THE 4-GONAL CONFIGURATION $G(T)$ OF ORDER $s = p^h$. Define points as (i) the s^3 elements of the group G (ii) the $s^2 + s$ cosets of the subgroups $H_1^*, H_2^*, \cdots, H_{s+1}^*$ (the cosets of H_i^* are denoted by $H_i^* = H_{i,1}^*, H_{i,2}^*, \cdots, H_{i,s}^*$ (iii) one new symbol x_∞. Define lines as (*a*) the $s^3 + s^2$ cosets of the subgroups $H_1, H_2, \cdots, H_{s+1}$ (*b*) the sets $L_i = \{H_{i,1}^*, H_{i,2}^*, \cdots, H_{i,s}^*\}$, $i = 1, 2, \cdots \cdots, s+1$. Incidence is defined as follows: Points of type (i) are incident only with lines of type (*a*); here the incidence relation is the natural incidence relation. A point $H_{i,j}^*$ of type (ii) is incident with all the cosets of type (*a*) which are subsets of the coset $H_{i,j}^*$ and with the line L_i of type (*b*). Finally, the unique point x_∞ of type (iii) is incident with all lines of type (*b*).

The configuration so defined is a tactical configuration $G(T) = (P, B, I)$ satisfying the following: $|P| = |B| = s^3 + s^2 + s + 1$; each point is incident with $s+1$ lines and two distinct points are incident with at most one line; each line is incident with $s+1$ points and two distinct lines are incident with at most one point. Moreover it is not difficult to prove that $G(T)$ does not possess triangles. Now a rather easy counting argument shows that $G(T) = (P, B, I)$ is a 4-gonal configuration of order s.

4. The equivalence of the 4-gonal configurations $G(T)$ and the translation 4-gonal configurations $S^{(x_\infty)}$

4.1. THEOREM. *The point x_∞ is a coregular point of the 4-gonal configuration* $G(T)$. *Moreover* $G(T) = G(T)^{(x_\infty)}$ *is a translation 4-gonal configuration for which the group of all translations is isomorphic to the group* G.

Proof. First of all we prove that x_∞ is a coregular point of $G(T)$. For that purpose we consider the line L_i, $i \in \{1, 2, \cdots, s+1\}$. A line which is not concurrent with L_i is of the form $H_{j,k}$, $i \neq j$. The $s+1$ lines which are concurrent with L_i and $H_{j,k}$ are the line L_j and the s cosets $H_{i,l_1}, H_{i,l_2}, \cdots, H_{i,l_s}$ of H_i which have an element in common with the coset $H_{j,k}$. We remark that $H_{i,l_1} \cup H_{i,l_2} \cup \cdots \cup H_{i,l_s}$ is a coset R of the group $H_i H_j$. This coset R contains also s cosets $H_{j,k_1} = H_{j,k}, H_{j,k_2}, \cdots, H_{j,k_s}$ of the subgroup H_j. Now we remark that the lines H_{j,k_t} $(t = 1, 2, \cdots, s)$, L_i are concurrent with the lines $H_{i,l_1}, H_{i,l_2}, \cdots, H_{i,l_s}, L_j$. There follows that the pair $(L_i, H_{j,k})$ is regular. Consequently the line L_i is regular, $i = 1, 2, \cdots, s+1$. So we conclude that the point x_∞ is coregular.

Now we consider the following bijection θ_a, $a \in G$, of the pointset P of $G(T)$ onto itself

(1) $x^{\theta_a} = ax$ for each point x of type (i);

(2) $(H^*_{i,j})^{\theta_a} = aH^*_{i,j}$ for each point $H^*_{i,j}$ of type (ii);

(3) $x_\infty^{\theta_a} = x_\infty$.

Evidently θ_a is a translation of the 4-gonal configuration $G(T)^{(x_\infty)}$. As the group $\{\text{all } \theta_a \,\|\, a \in G\} \cong G$ is transitive on the points of type (i) we conclude that $G(T)^{(x_\infty)}$ is a translation 4-gonal configuration for which the group of all translations is isomorphic to G.

4.2. REMARK. The translation plane of order s which corresponds with H_i (see 3.3.) evidently is isomorphic to the translation plane $\Pi^{L_\infty}_{L_i}$, with $L_\infty = \{L_1, L_2, \cdots, L_{s+1}\}$.

4.3. THEOREM. *Let* $S^{(x_\infty)} = (P, B, I)$ *be a translation 4-gonal configuration with coregular point* x_∞. *If* G *is the group of all translations of* $S^{(x_\infty)}$, *then* $S^{(x_\infty)}$ *is isomorphic to a 4-gonal configuration* $G(T)$.

Proof. First of all we remark that we use the notations of 2.

Let o be a point of $S^{(x_\infty)}$ which is not collinear with x_∞. Now we define the following bijection ω of $P \setminus P_\infty$ onto G:

$$\omega : P \setminus P_\infty \to G, \; x \to \theta \Longleftrightarrow o^\theta = x.$$

Next we consider a line L of $S^{(x_\infty)}$ which is incident with o and we suppose that L is concurrent with L_i. Then from 2.2. and 2.7. there follows that the set of points of $P \setminus P_\infty$ which are incident with L is mapped by ω onto the kernel H_i of the epimorphism Φ_i. In this way we obtain $s+1$ subgroups $H_1, H_2, \cdots, H_{s+1}$ of order s of the elementary abelian group G of order s^3

(remark that $H_i \cap H_j = \{1\}$, $i \neq j$ (1)). We shall prove that $H_i H_j H_k = G$, $\forall i, j, k$ with i, j, k distinct.

For that purpose we have to show that $H_i H_j \cap H_k = \{1\}$ (i, j, k distinct). Taking account of (1) it is sufficient to prove that $\theta\theta' \notin H_k$, $\forall \theta \in H_i \setminus \{1\}$ and $\forall \theta' \in H_j \setminus \{1\}$. Suppose a moment that $\theta\theta' \in H_k$, $\theta \in H_i \setminus \{1\}$ and $\theta' \in H_j \setminus \{1\}$ (evidently $\theta\theta' \neq 1$). If $o^\theta = x$ then o, x are collinear, and the line defined by o and x is concurrent with L_i; if $x^{\theta'} = x'$ then x, x' are collinear, and the line defined by x and x' is concurrent with L_j (this follows from the fact that θ' belongs to the kernel of Φ_j). Now from $o^{\theta\theta'} = x'$ and $\theta\theta' \in H_k$ there follows that also o, x' are collinear and that the line defined by o and x' is collinear with L_k. So we obtain a triangle (with vertices o, x, x') in $S^{(x_\infty)}$, a contradiction. Consequently $H_i H_j H_k = G$, $\forall i, j, k$ with i, j, k distinct. We conclude that $T = \{H_1, H_2, \cdots, H_{s+1}\}$ is a T-set of the abelian group G.

If $L \notin L_\infty$ is a line of $S^{(x_\infty)}$ which is concurrent with L_i, then the s points of $P \setminus P_\infty$ which are incident with L are mapped by ω onto the points of a coset of H_i. In this way we obtain the $s^3 + s^2$ cosets of the subgroups $H_1, H_2, \cdots, H_{s+1}$ of G.

Now we consider the point $x_i \mathrm{I} L_i$ which is collinear with o ($i = 1, 2, \cdots \cdots, s+1$). Notations are chosen in such a way that $x_i = x_{i,1}$. The s^2 points of $P \setminus P_\infty$ which are collinear with x_i are mapped by ω onto the points of the subgroup $H_i^* = H_{i,1}^*$ of G (see 3.1. and 3.2.). We remark that H_i^* is the stabilizer G_{x_i}. In this way we obtain the $s+1$ subgroups $H_1^*, H_2^*, \cdots, H_{s+1}^*$ of G. Next let $x_{i,j} \neq x_\infty$ be an arbitrary point which is incident with L_i. It is not difficult to show that the s^2 points of $P \setminus P_\infty$ which are collinear with $x_{i,j}$ are mapped by ω onto the points of a coset $H_{i,j}^*$ of H_i^* (we remark that $H_{i,j}^* = \{\text{all } \theta \in G \parallel x_i^\theta = x_{i,j}\}$). In this way we obtain the $s^2 + s$ cosets of the subgroups $H_1^*, H_2^*, \cdots, H_{s+1}^*$ of G.

Finally we define the following bijection ω^* of the pointset P of $S^{(x_\infty)}$ onto the pointset of the 4-gonal configuration $G(T)$:

(a) $x_\infty^{\omega^*} = x_\infty$;

(b) $x_{i,j}^{\omega^*} = H_{i,j}^*$, $\quad i = 1, 2, \cdots, s+1 \quad , \quad j = 1, 2, \cdots, s$;

(c) $x^{\omega^*} = x^\omega \ \forall x \in P \setminus P_\infty$.

As ω^* defines an isomorphism of the translation 4-gonal configuration $S^{(x_\infty)}$ onto the 4-gonal configuration $G(T)$, our theorem is completely proved.

5. THE KERNEL OF A TRANSLATION 4-GONAL CONFIGURATION

5.1. DEFINITION. Consider the translation 4-gonal configuration $S^{(x_\infty)}$ of order s (≥ 2) and the corresponding 4-gonal configuration $G(T)$. If $T = \{H_1, H_2, \cdots, H_{s+1}\}$ and $s > 2$, then the kernel of $S^{(x_\infty)}$ is the set F of all endomorphisms α of the group G with $H_i^\alpha \subseteq H_i$, $i = 1, 2, \cdots, s+1$; if $s = 2$, then the kernel of $S^{(x_\infty)}$ is the set $F = \{$identity automorphism of G, null endomorphism of G$\}$. With the usual addition and multiplication of endomorphisms F evidently is a ring.

5.2. THEOREM. *The kernel* F *of the translation* 4*-gonal configuration* $S^{(x_\infty)}$ *is a field.*

Proof. (a) If $s = 2$ then F evidently is a field.

(b) $s > 2$. Let α be an endomorphism of the group G of all translations of $S^{(x_\infty)}$ with $H_i^\alpha \subseteq H_i$, $i = 1, 2, \cdots, s+1$. Then α induces an endomorphism α_i of the group G/H_i, with $(H_j^{\sigma_i})^{\alpha_i} = H_j'^{\alpha_i} \subseteq H_j'$, $i \in \{1, 2, \cdots, s+1\}$ and $j \neq i$ (we use the notations of 3.2.). Now we shall prove that $H_i'^{\alpha_i} \subseteq H_i'$.

First of all we remark that $H_j'^{\alpha_i} \cap H_k'^{\alpha_i} = \{1'\}$, $k \neq i$, $j \neq i$, $k \neq j$. Let us suppose a moment that a'^{α_i} / H_i', with $a' \in H_i' \setminus \{1'\}$. Then $a'^{\alpha_i} = b' \in H_j' \setminus \{1'\}$, with $j \neq i$. Now we choose an arbitrary element $b'' \in H_j' \setminus \{1'\}$. Then there exist a $k \in \{1, 2, \cdots, s+1\} \setminus \{i, j\}$ and an element $c' \in H_k' \setminus \{1'\}$, such that $a' = b''c'$ {see proof of 3.2.). Consequently $b' = a'^{\alpha_i} = b''^{\alpha_i} c'^{\alpha_i}$ or $b'(b''^{-1})^{\alpha_i} = c'^{\alpha_i}$. As $b'(b''^{-1})^{\alpha_i} \in H_j'$ and $c'^{\alpha_i} \in H_k'$, there follows that $c'^{\alpha_i} = 1'$ and $b'(b''^{-1})^{\alpha_i} = 1'$. So $b' = b''^{\alpha_i}$ $\forall b'' \in H_j' \setminus \{1'\}$. Since α_i induces an endomorphism of the group H_j', we have necessarily $|H_j' \setminus \{1'\}| = 1$ or $|H_j'| = s = 2$, a contradiction. So we conclude that $H_i'^{\alpha_i} \subseteq H_i'$.

From $H_i'^{\alpha_i} \subseteq H_i'$ and $(H_j^{\sigma_i})^{\alpha_i} \subseteq H_j^{\sigma_i}$, $j \neq i$, there follows that α_i belongs to the kernel of the translation plane defined by H_i, and so α_i is the null endomorphism of G/H_i or an automorphism of G/H_i [2]. If F_i is the kernel of the translation plane defined by H_i (i.e. the translation plane $\Pi_{L_i}^{L_\infty}$ (see 4.2.)), then $\Delta_i : F \to F_i$, $\alpha \to \alpha_i$ evidently is a homomorphism of the ring F into the ring $F_i (i = 1, 2, \cdots, s+1)$. Now we shall prove that Δ_i is a monomorphism $(i = 1, 2, \cdots, s+1)$.

Suppose that α_i is the null endomorphism of G/H_i and that $\Delta_i(\alpha) = \alpha_i$. Then we have $a^\alpha \in H_i$ $\forall a \in G$, and so $a^\alpha = 1$ $\forall a \in H_1 \cup \cdots \cup H_{i-1} \cup H_{i+1} \cup \cdots \cup H_{s+1}$. Now we consider $\alpha_j = \Delta_j(\alpha)$, $j \neq i$. If $a \in H_k \setminus \{1\}$, $k \notin \{i, j\}$, then $a^\alpha = 1$ and so $(aH_j)^{\alpha_j} = H_j$. Consequently α_j, $j \neq i$, is the null endomorphism of G/H_j. There follows that $a^\alpha \in H_j$, $\forall a \in G$ and $\forall j \in \{1, 2, \cdots, s+1\}$. Hence $a^\alpha = 1$ $\forall a \in G$, and this means that α is the null endomorphism of G. So we conclude that Δ_i is a monomorphism. Finally we prove that α is the null endomorphism of G or an automorphism of G.

We suppose that $a^\alpha = 1$, $a \in G \setminus \{1\}$. When $a \notin H_i$, then $(aH_i)^{\alpha_i} = H_i$ and consequently α_i is the null endomorphism of G/H_i. From the preceding there follows immediately that α is the null endomorphism of G. We conclude that any element $\alpha \in F$ is the null endomorphism of G or an automorphism of G, and so the ring F is a field.

5.3. THEOREM. *The kernel* F *of the translation* 4*-gonal configuration* $S^{(x_\infty)}$ *of order* $s = p^h$ *is a subfield of the kernel* F_i *of the translation plane* $\Pi_{L_i}^{L_\infty}$ *(of order* $s = p^h$*),* $i = 1, 2, \cdots, s+1$. *Consequently* $|F| \leq s$ *and* $|F| = p^{h'}$ $(1 \leq h' \leq h)$ [2].

Proof. (a) If $s = 2$, then $F = F_i = GF(2)$.

(b) $s > 2$. In this case the theorem follows immediately from the fact that Δ_i is a monomorphism of the field F into the field F_i.

5.4. THE VECTOR SPACE G. The group G may be regarded as the additive group of a vector space over any subfield F′ of the kernel F of $S^{(x_\infty)}$. This vector space is also denoted by G and its dimension is denoted by $[G : F']$. We remark that $[G : F'] \geq 3$ (if $[G : F'] = 3$ then necessarily $F = F'$).

5.5. THEOREM. *We have* $[G : F'] = 3n$, $n \geq 1$.

Proof. The subgroups H_1, H_2, H_3 of the group G may be regarded as subspaces of the vector space G over F′. As $|H_1| = |H_2| = |H_3| = s$, we have $[H_1 : F'] = [H_2 : F'] = [H_3 : F'] = n\,(n \geq 1)$. From $H_1 H_2 H_3 = G$ and $|H_1|\,|H_2|\,|H_3| = |G|$, there follows immediately that $[G : F'] = [H_1 : F'] + [H_2 : F'] + [H_3 : F'] = 3n$.

5.6. DESARGUESIAN TRANSLATION 4-GONAL CONFIGURATIONS. The translation 4-gonal configuration $S^{(x_\infty)}$ is called desarguesian if $[G : F] = 3$ (i.e. if $|F| = s$). If $S^{(x_\infty)}$ is desarguesian then $|F_i| = s$, and consequently the translation plane $\Pi^{L_\infty}_{L_i}$ is desarguesian $(i = 1, 2, \cdots, s+1)$ [2].

6. THE 4-GONAL CONFIGURATIONS $T(n, q)$

6.1. THE 4-GONAL CONFIGURATIONS $T(n, q)$. In $PG(3n-1, q)$, q a prime power and $n \geq 1$, we consider $q^n + 1$ $(n-1)$-dimensional subspaces $PG^{(1)}(n-1, q)$, $PG^{(2)}(n-1, q), \cdots, PG^{(q^n+1)}(n-1, q)$, every three of them being joined by $PG(3n-1, q)$ (with such a set of subspaces there corresponds a (q^n+1)-arc K of the projective plane over the total matrix algebra of the $n\times n$-matrices with elements in $GF(q)$ [8]). In [8] we have proved that through $PG^{(i)}(n-1, q)$, $i = 1, 2, \cdots, q^n+1$, there passes one and only one subspace $PG^{(i)}(2n-1, q)$ of $PG(3n-1, q)$ which has no point in common with the set $PG^{(1)}(n-1, q) \cup \cdots \cup PG^{(i-1)}(n-1, q) \cup PG^{(i+1)}(n-1, q) \cup \cdots \cup PG^{(q^n+1)}(n-1, q)$ (with the q^n+1 spaces $PG^{(i)}(2n-1, q)$ there correspond the q^n+1 tangent lines of the (q^n+1)-arc K [8]).

Let $PG(3n-1, q)$ be embedded as a hyperplane H_∞ in $PG(3n, q) = P$. Define points of the incidence structure $T(n, q)$ as (i) the points of $P \setminus H_\infty$ (ii) the $2n$-dimensional subspaces X of P for which $X \cap H_\infty = PG^{(i)}(2n-1, q)$, $i \in \{1, 2, \cdots, q^n+1\}$ (iii) one new symbol x_∞. Lines of the configuration are (*a*) the n-dimensional subspaces of P which are not contained in H_∞ and pass through one of the spaces $PG^{(1)}(n-1, q)$, $PG^{(2)}(n-1, q), \cdots, PG^{(q^n+1)}(n-1, q)$, and (*b*) the spaces $PG^{(1)}(n-1, q)$, $PG^{(2)}(n-1, q), \cdots, PG^{(q^n+1)}(n-1, q)$. Incidence is defined as follows: Points of type (i) are incident only with lines of type (*a*); here the incidence is that of P. A point X of type (ii) is incident with all lines $\subset X$ of type (*a*) and with precisely one line of type (*b*), namely the one represented by the unique space $PG^{(i)}(n-1, q)$ in X. Finally, the unique point x_∞ of type (iii) is incident with no line of type (*a*) and all lines of type (*b*).

The incidence structure $T(n, q)$ so defined is a 4-gonal configuration of order q^n [9].

6.2. THE 4-GONAL CONFIGURATIONS $T(1, q)$ OF J. TITS. For $n = 1$ we obtain 4-gonal configurations $T(1, q)$, of order q, arising from $(q+1)$-arcs in $PG(2, q)$. Consequently the configurations $T(1, q)$ are the 4-gonal configurations (of order q) constructed by J. Tits [2].

7. THE EQUIVALENCE OF THE 4-GONAL CONFIGURATIONS $T(n, q)$ AND THE TRANSLATION 4-GONAL CONFIGURATIONS $S^{(x_\infty)}$

7.1. THEOREM. *The point x_∞ is a coregular point of the 4-gonal configuration $T(n, q)$. Moreover $T(n, q) = T(n, q)^{(x_\infty)}$ is a translation 4-gonal configuration for which the group G of all translations is isomorphic to the group of all translations of the affine space $AG(3n, q) = PG(3n, q)^{H_\infty}$. Finally the field $GF(q) = F'$ is a subfield of the kernel F of $T(n, q)^{(x_\infty)}$, and $[G : F'] = 3n$.*

Proof. First of all we prove that x_∞ is a coregular point of $T(n, q)$. For that purpose we consider the line $PG^{(i)}(n-1, q) = L_i$, $i \in \{1, 2, \cdots, q^n+1\}$. A line which is not concurrent with L_i is of the form $PG(n, q)$ with $PG(n, q) \not\subset H_\infty$ and $PG^{(j)}(n-1, q) \subset PG(n, q)$ $(i \neq j)$. Let $PG(2n, q)$ denote the $2n$-dimensional projective space joining $PG^{(i)}(n-1, q)$ and $PG(n, q)$. The q^n+1 lines which are concurrent with L_i and $PG(n, q)$ are the line $L_j = PG^{(j)}(n-1, q)$ and the q^n lines $PG^{(l_1)}(n, q)$, $PG^{(l_2)}(n, q), \cdots, PG^{(l_{q^n})}(n, q)$ (of type (*a*)), for which $PG^{(l_t)}(n, q) \subset PG(2n, q)$ and $PG^{(i)}(n-1, q) \subset PG^{(l_t)}(n, q)$ $(t = 1, 2, \cdots, q^n)$. Next let $PG^{(k_1)}(n, q) = PG(n, q)$, $PG^{(k_2)}(n, q), \cdots, PG^{(k_{q^n})}(n, q)$ be the q^n lines (of type (*a*)) for which $PG^{(j)}(n-1, q) \subset PG^{(k_t)}(n, q)$ and $PG^{(k_t)}(n, q) \subset PG(2n, q)$ $(t = 1, 2, \cdots, q^n)$. Now we remark that the lines $PG^{(k_t)}(n, q)$ $(t = 1, 2, \cdots, q^n)$ and L_i are concurrent with the lines $PG^{(l_1)}(n, q)$, $PG^{(l_2)}(n, q), \cdots, PG^{(l_{q^n})}(n, q)$, L_j. There follows immediately that the pair $(L_i, PG(n, q))$ is regular. Consequently the line L_i is regular, $i = 1, 2, \cdots, q^n+1$. So we conclude that the point x_∞ is coregular.

Next we remark that each translation of the affine space $AG(3n, q) = PG(3n, q)^{H_\infty}$ induces a translation of the 4-gonal configuration $T(n, q)^{(x_\infty)}$. As the group of all translations of $AG(3n, q)$ is transitive on the points of $AG(3n, q)$ (i.e. on the points of $T(n, q)^{(x_\infty)}$ which are not collinear with x_∞), there follows immediately that $T(n, q)^{(x_\infty)}$ is a translation 4-gonal configuration for which the group G of all translations is isomorphic to the translation group of the affine space $AG(3n, q)$.

Now we observe that it is easy to prove that the field $GF(q) = F'$ is a subfield of the kernel F of $T(n, q)^{(x_\infty)}$ (the multiplicative group $F' \setminus \{$null endomorphism of G } corresponds with (and is isomorphic to) the group of all dilatations of $AG(3n, q)$ with center $o \notin H_\infty$ (see 4.3.)). As $|G| = q^{3n}$ and $|F'| = q$, we have $[G : F'] = 3n$, and so the theorem is completely proved.

7.2. THEOREM. *Let* $S^{(x_\infty)}$ *be a translation 4-gonal configuration (of order* s*) with coregular point* x_∞ *and translation group* G. *If* $GF(q) = F'$ *is a subfield of the kernel* F *of* $S^{(x_\infty)}$, *where* $[G : F'] = 3n$, *then* $S^{(x_\infty)}$ *is isomorphic to a 4-gonal configuration* $T(n, q)$.

Proof. We consider the 4-gonal configuration $G(T)$, $T = \{H_1, H_2, \cdots, H_{s+1}\}$ $(s = q^n)$, defined by $S^{(x_\infty)}$. With the vector space G over $GF(q) = F'$ there corresponds a $3n$-dimensional affine space $AG(3n, q)$ (the subspaces of $AG(3n, q)$ are the cosets of the subgroups of the group G). If H_∞ is the ideal hyperplane of $AG(3n, q)$, then $AG(3n, q) = PG(3n, q)^{H_\infty}$, with $PG(3n, q)$ the $3n$-dimensional projective space over $GF(q)$ defined by $AG(3n, q)$.

With the s^2 cosets of the group H_i there correspond the q^{2n} n-dimensional subspaces of $PG(3n, q)$ which are not contained in H_∞ and which pass through a certain $PG^{(i)}(n-1, q) \subset H_\infty$ $(i = 1, 2, \cdots, s+1 = q^n + 1)$. As the n-dimensional subspaces H_i, H_j, H_k $(i, j, k$ distinct) of the affine space $AG(3n, q)$ are joined by $AG(3n, q)$, the projective spaces $PG^{(i)}(n-1, q)$, $PG^{(j)}(n-1, q)$, $PG^{(k)}(n-1, q)$ are joined by the $(3n-1)$-dimensional projective space H_∞.

With the s cosets of the group H_i^* (see 3.) there correspond the q^n $2n$-dimensional subspaces of $PG(3n, q)$ which are not contained in H_∞ and which pass through a certain $PG^{(i)}(2n-1, q) \subset H_\infty$ $(i = 1, 2, \cdots, q^n + 1)$. First of all we remark that $PG^{(i)}(n-1, q)$ is a subspace of $PG^{(i)}(2n-1, q)$ (this follows from $H_i \subset H_i^*$). As the subspaces H_i^*, H_j $(i \neq j)$ of the affine space $AG(3n, q)$ are joined by $AG(3n, q)$, the projective spaces $PG^{(i)}(2n-1, q)$, $PG^{(j)}(n-1, q)$ $(i \neq j)$ are joined by H_∞. Consequently $PG^{(i)}(2n-1, q) \cap PG^{(j)}(n-1, q) = \varnothing$ $(i \neq j)$. So the space $PG^{(i)}(2n-1, q)$ has no point in common with the set $PG^{(1)}(n-1, q) \cup \cdots \cup PG^{(i-1)}(n-1, q) \cup \cup PG^{(i+1)}(n-1, q) \cup \cdots \cup PG^{(q^n+1)}(n-1, q)$.

Now we see immediately that the 4-gonal configuration $T(n, q)$, defined by the subspaces $PG^{(1)}(n-1, q)$, $PG^{(2)}(n-1, q), \cdots, PG^{(q^n+1)}(n-1, q)$ of the $(3n-1)$-dimensional projective space H_∞, is isomorphic to the 4-gonal configuration $G(T)$. We conclude that $S^{(x_\infty)}$ is isomorphic to $T(n, q)$.

7.3. THEOREM. *The translation 4-gonal configuration* $S^{(x_\infty)}$ *is desarguesian if and only if it is isomorphic to a 4-gonal configuration* $T(1, q)$ *of Tits.*

Proof. Let $S^{(x_\infty)}$ be a desarguesian translation 4-gonal configuration of order $p^h = q$. Then $[G : F] = 3$ (i.e. $n = 1$) and $F = GF(q)$. From 7.2. there follows immediately that $S^{(x_\infty)}$ is isomorphic to a 4-gonal configuration $T(1, q)$ of Tits.

Conversely, let us suppose that the translation 4-gonal configuration $S^{(x_\infty)}$ is isomorphic to a 4-gonal configuration $T(1, q)$ of Tits. Then $|G| = q^3$ and $|F| \geq q$ (see 7.1.). As q is the order of $S^{(x_\infty)}$ there holds $|F| \leq q$ (5.3.), and so $|F| = q$. Consequently $[G:F] = 3$, i.e. $S^{(x_\infty)}$ is desarguesian.

REFERENCES

[1] J. ANDRÉ, *Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe*, «Math. Zeitschr.», *60*, 156–186 (1954).

[2] P. DEMBOWSKI, *Finite geometries*, Springer–Verlag, 1968, 375 pp.

[3] S. E. PAYNE, *Affine representations of generalized quadrangles*, « J. Algebra », *16*, 473–485 (1970).

[4] S. E. PAYNE, *Generalized quadrangles as amalgamations of projective planes*, « J. Algebra », *22*, 120–136 (1972).

[5] S. E. PAYNE, *Generalized quadrangles of even order*, 35 pp. (to appear).

[6] R. R. SINGLETON, *Minimal regular graphs of maximal even girth*, « J. Comb. Theory », *1*, 306–332 (1966).

[7] J. A. THAS, *Een studie betreffende de projectieve rechte over de totale matrix algebra* $M_3(K)$ *der* 3×3*-matrices met elementen in een algebraïsch afgesloten veld* K, « Verh. Kon. Vl. Acad. Wet., Lett. Sch. K. van België, Kl. der Wet. », *31* (112), 151 pp. (1969).

[8] J. A. THAS, *The m-dimensional projective space* $S_m(M_n(GF(q)))$ *over the total matrix algebra* $M_n(GF(q))$ *of the* $n \times n$*-matrices with elements in the Galois field* $GF(q)$, « Rend. di Mat. », (6) *4*, 459–532 (1971).

[9] J. A. THAS, *On 4-gonal configurations*, «Geometriae Dedicata», 15 pp. (to appear).

[10] J. A. THAS, *4-gonal configurations*, C.I.M.E. Session Bressanone, 249–263 (1972).