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Theorems of the Liouville type for elliptic systems of partial differential equations

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Analisi matematica. — *Theorems of the Liouville type for elliptic systems of partial differential equations.* Nota di OLGA A. OLEINIK, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si considerano sistemi ellittici generali di equazioni alle derivate parziali e per essi si stabiliscono teoremi analoghi a quello classico di Liouville sulle funzioni armoniche, poggiando su di una proposizione relativa all'analiticità delle soluzioni di certe equazioni ellittiche. Alcuni teoremi del tipo di quelli qui ottenuti trovansi già in [1].

This paper has been the subject of a lecture given at the «Centro Linceo Interdisciplinare di Scienze matematiche e loro applicazioni ». In it theorems—similar to the Liouville theorem on harmonic functions—are proved for general elliptic systems of partial differential equations, on the bases of a theorem about the analyticity of solutions of elliptic equations. Some theorems of this type were already given in [1].

Let ω be a domain in $R^n = (x_1, \cdots, x_n)$, $\Omega = \omega \times \{|x_0| < T\}$ and $\Omega \subset R^{n+1} = (x_0, x_1, \cdots, x_n) \equiv (x_0, x)$. Let us consider in Ω a system of partial differential equation of the form

$$(1) \qquad \sum_{j=1}^{N} \sum_{\alpha_0 + |\alpha| \leq m_{kj}} \mathcal{A}^{kj}_{\alpha_0 \alpha} \mathcal{D}^{\alpha_0}_{x_0} \mathcal{D}^{\alpha}_{x} u_j = 0, \qquad k = 1, \cdots, N,$$

where

$$\alpha = (\alpha_1, \cdots, \alpha_n) \quad , \quad |\alpha| = \alpha_1 + \cdots + \alpha_n ,$$

$$\mathcal{D}_{x_0} = \frac{\partial}{\partial x_0} \quad , \quad \mathcal{D}_{x_s} = \frac{\partial}{\partial x_s} \quad , \quad \mathcal{D}^{\alpha}_{x} = \mathcal{D}^{\alpha_1}_{x_1} \cdots \mathcal{D}^{\alpha_n}_{x_n} .$$

We suppose that system (1) is elliptic in Ω; this means that there exist two sets of integers $s_1, \cdots, s_N$ and $t_1, \cdots, t_N$ such that for any k and $j\,(k, j = 1, \cdots, N)$, we have

$$m_{kj} \leq s_k + t_j \quad , \quad \sum_{j=1}^{N} s_j + t_j = m \quad , \quad m_{kj} = 0, \qquad \text{if} \quad s_k + t_j \leq 0,$$

and, for any vector $(\xi_0, \xi_1, \cdots, \xi_n) \equiv (\xi_0, \xi) \subset R^{n+1}$ with $|\xi_0| + |\xi| \neq 0$ and for any $(x_0, x) \subset \Omega$.

$$P_m(x_0, x, \xi_0, \xi) \equiv \det \left\| \sum_{\alpha_0 + |\alpha| = s_k + t_j} \mathcal{A}^{kj}_{\alpha_0 \alpha} \xi_0^{\alpha_0} \xi^{\alpha} \right\| \neq 0,$$

where $\xi^{\alpha} \equiv \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. We can suppose that $s_k \leq 0$ and $t_j \geq 0$.

(*) Nella seduta del 9 marzo 1974.

System (1) is said to be uniformly elliptic in Ω if, for any $(\xi_0, \xi) \in R^{n+1}$ and $(x_0, x) \in \Omega$, the inequality

$$\lambda (|\xi_0|^2 + |\xi|^2)^{m/2} \leq |P_m(x_0, x, \xi_0, \xi)| \leq \lambda^{-1} (|\xi_0|^2 + |\xi|^2)^{m/2}, \lambda = \text{const.}$$

holds.

Let us consider in ω an elliptic system of partial differential equations of the form

$$\sum_{j=1}^{N} \sum_{\alpha_0 + |\alpha| \leq m_{kj}} \mathfrak{A}^{kj}_{\alpha_0 \alpha}(x) \mu^{\alpha_0} \mathfrak{D}^{\alpha}_x u_j = 0, \qquad k = 1, \cdots, N, \tag{2}$$

$\mu \in R^1$. We define the norm

$$\|f\|_{C^M(\omega)} = \sum_{|\alpha| \leq M} \sup_{\omega} |\mathfrak{D}^{\alpha} f|.$$

LEMMA 1. *Suppose that the uniformly elliptic system* (1) *is defined in a domain* $\Omega = \omega \times \{|x_0| < T\}$ *and* $\mathfrak{A}^{kj}_{\alpha_0 \alpha}(x) \in C^{-s_j+\gamma+1}(\omega)$, $\gamma =$ *const.* ≥ 0. *Then any solution* $(u_1, \cdots, u_N)$ *of system* (1) *such that* $u_j \in C^{t_j+1}(\Omega)$ *can be extended as an analytic function of* $x_0 + iy_0$ *in the domain*

$$Q_\delta(G) = \{x_0, x, y_0; (x_0, x) \in G, |y_0| < \delta\}$$

where $G = g \times \{|x_0| < T - 1\}$, $g \subset R^n$ *and* $\bar{g} \subset \omega$, $\delta =$ *const.* > 0. *In addition, the derivatives of* u_j *up to order* $t_j + \gamma - 1$ *can be extended as analytic functions of* $x_0 + iy_0$ *in* $Q_\delta(G)$ *and the inequality*

$$\sup_{Q_\delta(G)} |\mathfrak{D}^{\alpha}_x u_j| \leq C \sup_{\Omega} |u|, \qquad j = 1, \cdots, N \tag{3}$$

is satisfied, where $|\alpha| \leq t_j + \gamma - 1$, *the constants* δ *and* C *depend on* λ, n, N, m, *the norms of* $\mathfrak{A}^{kj}_{\alpha_0 \alpha}$ *in* $C^{-s_j+\gamma+1}(\omega)$ *and the distance between* G *and* Ω *only.*

THEOREM 1. *Suppose that system* (1) *is uniformly elliptic in* R^{n+1}, $\mathfrak{A}^{kj}_{\alpha_0 \alpha}(x) \in C^{-s_j+1}(R^n)$. *Let* $u(x) = (u_1(x), \cdots, u_N(x))$ *be a solution of system* (2), $u_j(x) \in C^{t_j+1}(\omega)$ *for any finite domain* ω. *Suppose that*

$$|u_j(x)| \leq M_1 \, exp \, \{\delta_1 |x|\} \ in \ R^n, \qquad j = 1, \cdots, N \tag{4}$$

M_1, δ_1 *are constants,* $\delta_1 > 0$. *Then*

$$u_j \equiv 0 \ in \ R^n, \qquad j = 1, \cdots, N,$$

if $\mu \geq \mu_0$ *and* $\delta_1 - \mu_0 \delta + \ln C < 0$, *where the constants* δ *and* C *depend on* λ, n, N, m *and the norms of* $\mathfrak{A}^{kj}_{\alpha_0 \alpha}$ *in* $C^{-s_j+1}(R^n)$ *only.*

Proof. Let us denote

$$\Omega_s = \{x_0, x; |x_0| < s, |x| < s\} \quad , \quad \omega_s = \{x; |x| < s\}.$$

It is easy to see that $w(x_0, x) = \exp\{i\mu x_0\} u(x)$ is a solution of system (1) in R^{n+1}. According to Lemma 1 the vector-function $w(x_0, x)$ is analytic with respect to $x_0 + iy_0$, and the estimate

(5) $$\sup_{Q_\delta(\Omega_s)} |w| \leq C \sup_{\Omega_{s+1}} |w|$$

is valid where the constants δ and C do not depend on s. From the estimate (5) it follows that

$$\sup_{\omega_s} |u| \leq C \exp\{-\mu\delta\} \sup_{\omega_{s+1}} |u| = \exp\{-\mu\delta + \ln C\} \sup_{\omega_{s+1}} |u|$$

and therefore

$$\sup_{\omega_{s_0}} |u| \leq \exp\{s(-\mu\delta + \ln C)\} \sup_{\omega_{s_0+s}} |u|.$$

Since (4) is valid, we have

$$\sup_{\omega_{s_0}} |u| \leq \exp\{s(-\mu\delta + \ln C\} M_1 \exp\{\delta_1(s_0 + s)\} =$$

$$= M_1 \exp\{\delta_1 s_0\} \cdot \exp\{s(-\mu\delta + \ln C + \delta_1)\} \equiv \varphi_s.$$

If $-\mu\delta + \ln C + \delta_1 < 0$, then $\varphi_s \to 0$ as $s \to \infty$ and therefore $u_j \equiv 0$, $j = 1, \cdots, N$.

THEOREM 2. *Let* ω *and* ω_0 *be unbounded domains in* R^n, $\bar{\omega}_0 \subset \omega$. *For* $x \in \omega_0$ *we define a function* $p(x)$ *which is equal to the distance from* x *to the boundary of* ω. *Suppose that* $p(x) \to \infty$ *as* $|x| \to \infty$, *system* (1) *is uniformly elliptic in* $\Omega = \omega \times \{-\infty < x_0 < +\infty\}$ *and* $\mathfrak{A}^{kj}_{\alpha_0\alpha} \in C^{-s_j+l+1}(\Omega)$, $l \geq 0$. *Then for any solution* $u(x) = (u_1(x), \cdots, u_N(x))$ *of system* (2)

$$\sum_{|\alpha| \leq t_j+l-1} |\mathfrak{D}^\alpha u_j(x)| \leq M \exp\{-\delta_0 p(x)\}$$

for any $x \in \omega_0$, *if* $\mu \geq \mu_0$, $u_j(x) \in C^{t_j+1}(g)$ *for any bounded subdomain* g *of* ω *and if*

(6) $$\sup_{S_{p(x)}} |u| \leq M_0 \exp\{\delta_1 p(x)\}$$

for any $x \in \omega_0$, *where* $M, M_0, \delta_0, \delta_1$ *are constants*, $-\delta\mu_0 + \ln C + \delta_0 + \delta_1 \leq 0$, $\delta_0, \delta_1 > 0$, $S_{p(x)}$ *is a ball of radius* $p(x)$ *with center in* x; δ *and* C *depend on* λ, n, N, l, m *and the norms* $\mathfrak{A}^{kj}_{\alpha_0\alpha}$ *in* $C^{-s_j+l+1}(\omega)$ *only.*

Proof. Let $\hat{x}$ be an arbitrary point in ω. Let us consider balls $g_1, g_2, \cdots, g_{[p(\hat{x})]}$ of radius $1, 2, \cdots, [p(\hat{x})]$ respectively, with center in $\hat{x}$. Here $[p(x)]$ denotes the entire part of $p(x)$. Let $G_s = g_s \times \{|x_0| < s\}$, $s = 1, \cdots, [p(\hat{x})\}$. If $u(x)$ is a solution of system (2) in ω, then

$$v(x_0, x) = \exp\{i\mu x_0\} u(x)$$

is a solution of system (1) in $\Omega = \omega \times \{-\infty < x_0 < +\infty\}$. According to estimate (3) we have

$$\sup_{Q_\delta(G_s)} \sum_{j=1}^{N} \sum_{|\alpha| \leq t_j + l - 1} |\mathfrak{D}_x^\alpha v_j| \leq C \sup_{G_{s+1}} |v| . \tag{7}$$

Since the solution $u(x)$ of system (2) satisfies condition (6), it follows from (7) that

$$\begin{aligned} \sup_{g_1} \sum_{|\alpha| \leq t_j + l - 1} |\mathfrak{D}_x^\alpha u_j| &\leq \exp\{-\delta\mu + \ln C\} \sup_{g_2} |u| \leq \\ &\leq \exp\{(-\delta\mu + \ln C)([p(\hat{x})] - 1)\} \sup_{g_{[p(x)]}} |u| \leq \\ &\leq \exp\{(-\delta\mu + \ln C)([p(\hat{x})] - 1) + \delta_1 p(\hat{x})\} M_0 \leq M \exp\{-\delta_0\, p(\hat{x})\} \end{aligned}$$

for any $\hat{x} \in \omega_0$. Thus the Theorem is proved.

Remark. Suppose that

$$\omega = \{\tau x \,;\, x \in g \,,\, \tau \geq 1\} \quad \text{and} \quad \omega_0 = \{\tau x \,;\, x \in g_0 \,,\, \tau \geq 1\},$$

where g is a bounded domain not containing the origin and g_0 is a subdomain of g, $\bar{g}_0 \subset g$. Then it is easy to see that

$$k_1 |x| \leq p(x) \leq k_2 |x| ,$$

where k_1 and k_2 are constants.

Similar theorems can be proved for parabolic systems in the sense of I. G. Petrovsky.

References

[1] O. A. Oleinik and E. V. Radkevich, *On a behaviour at the infinity of solutions of some systems of partial differential equations*, «Uspechi Mat. Nauk», 67 (5), 249–250 (1973).