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Contraction property of the operator of integration

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 9 marzo 1974

Presiede il Presidente della Classe BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Contraction property of the operator of integration.* Nota di LUDVIK JANOS, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si prova che l'operatore di integrazione $Fy(x) = \int_0^x y(t)\,dt$ definito sullo spazio $C(-\infty, \infty)$ delle funzioni reali continue su $(-\infty, \infty)$ è una contrazione rispetto ad una certa famiglia di seminorme che generano la topologia della convergenza uniforme sui compatti. Tuttavia, si prova anche, per contro, che F non è contrattiva rispetto ad alcuna metrica su $C(-\infty, \infty)$ che induca su $C(-\infty, \infty)$ la topologia suddetta.

1. INTRODUCTION

Let X be a metrizable topological space and $F: X \to X$ a continuous selfmapping of X into itself. We say F is a *topological contraction* if there is a suitable metric ρ on X inducing the topology of X and a constant $q \in (0, 1)$ such that $\rho(Fx, Fy) \leq q\rho(x, y)$ for all $x, y \in X$.

Assume now X is a Fréchet linear topological space and $F: X \to X$ a linear operator on X satisfying the following condition:

There exists a sequence of seminorms $\{p_n \mid n \geq 1\}$ on X inducing the topology of X and a number $q \in (0, 1)$ such that $p_n(Fx) \leq q p_n(x)$ for all $x \in X$ and all $n = 1, 2, \cdots$. It is natural to call such a linear operator F a *generalized contraction* on X. In [1] has been investigated a more general case where X is a completely regular not necessarily metrizable topological space and $F: X \to X$ a contraction with respect to a suitable family of pseudometrics inducing the topology of X.

Since the Fréchet space X is a metrizable topological space a question arises whether a generalized contraction on X is also a topological contrac-

(*) Nella seduta del 9 marzo 1974.

tion in the sense of the first definition. The main purpose of this note is to show that the answer is " no ", exhibiting at the same time a contraction property of the operator of integration $y(x) \to \int_0^x y(t)\,dt$ in the Fréchet space $C(-\infty, \infty)$. We prove the following.

THEOREM. *Let* $C = C(-\infty, \infty)$ *denote the linear space of all continuous real valued functions on* $(-\infty, \infty)$ *endowed with the topology of uniform convergence on compacta, and let* $F : C \to C$ *be defined by* $Fy(x) = \int_0^x y(t)\,dt$ *for* $y \in C$. *Then the operator* F *is a generalized contraction on* C *but it is not a topological contraction on* C.

2. PROOF OF THE THEOREM

LEMMA. *Let* X *be a metrizable topological space and* $F : X \to X$ *a self-mapping on* X *such that the following conditions are satisfied*:

(*i*) *there is a fixed point* $x_0 \in X$ *of* F, *i.e.*, $F(x_0) = x_0$;

(*ii*) *there is a metric* ρ *on* X *inducing the topology of* X *relative to which* F *is a contraction, i.e., there exists a constant* $q \in (0, 1)$ *such that* $\rho(Fx, Fy) \leq q\rho(x, y)$ *for all* $x, y \in X$.

Then there exists an open neighbourhood $U(x_0)$ *of* $\{x_0\}$ *such that for any neighbourhood* $V(x_0)$ *of* $\{x_0\}$ *there is an integer* $k_0 \geq 1$ *for which the following implication holds*: $k \geq k_0 \Rightarrow F^k(U(x_0)) \subset V(x_0)$, *showing that the iterated images* $F^k(U(x_0))$ *of* $U(x_0)$ *under* F *shrink into any prescribed neighbourhood* $V(x_0)$ *of* x_0 *for sufficiently large values of* k.

Proof. Let ρ be the metric on X satisfying the condition (ii) and $q \in (0, 1)$, the corresponding Lipschitz constant.

If we define $U(x_0)$ by $U(x_0) = \{x \in X : \rho(x, x_0) < 1\}$ we obtain for the diameters $\delta(F^k U(x_0))$ of the iterated images $F^k(U(x_0))$ the relation $\delta F^k(U(x_0)) \leq 2q^k \to 0$ as $k \to \infty$, showing that for any neighbourhood $V(x_0)$ of x_0 the sets $F^k(U(x_0))$ shrink into $V(x_0)$ if k is sufficiently large which accomplishes the proof.

We are now in the position to prove our theorem. First of all we observe that the topology of C can be induced by the sequence of seminorms defined by

$$\sup_{-n \leq x \leq n} |f(x)| \qquad \text{for any} \quad m = 1, 2, \cdots, \qquad \text{and} \quad f \in C.$$

However, the operator F is not contractive with respect to this family. By methods developed by S. C. Chu and J. B. Diaz for Banach spaces ([2]) one finds easily that the equivalent family $\{p_n \mid n \geq 1\}$ of seminorms defined by

$$p_n(f) = \sup_{-n \leq x \leq n} e^{-2|x|} |f(x)|$$

for $f \in C$ and $n = 1, 2, \cdots$ satisfies the relations

$$p_n(Fy) \leq \frac{1}{2} p_n(y)$$

for all $n = 1, 2, \cdots$ and $y \in C$, proving thus that F is a generalized contraction.

Suppose now that our operator $F : C \to C$ is a topological contraction. As the constant $0 \in C$ is the fixed point of F it follows that F would satisfy the conditions of our Lemma for some metric ρ inducing the topology of C. Let $U(0)$ be the neighbourhood of $\{0\}$ in C existing according to the Lemma and consider the fundamental system of neighbourhoods $\{U(n, a) \mid n \geq 1, a > 0\}$ of $\{0\}$ defined by

$$U(n, a) = \{f \in C : p_n(f) < a\}.$$

It follows that there is some $n \geq 1$ and $a > 0$ such that $U(n, a) \subset U(0)$ so that the neighbourhood $U(n, a)$ also would satisfy the conclusion of our Lemma. Choosing $V(0)$ to be $U(n+1, 1)$ we consider the function $y_n \in C$ defined by $y_n(x) = 0$ for $x \leq n$ and $y_n(x) = x - n$ for $x > n$. Then obviously $b \cdot y_n \in U(n, a)$ for any constant b but on the other hand for every $x > n$ and any $k \geq 1$ we have $F^k y_n(x) > 0$. Thus for any k we can choose b_k in such a way that

$$b_k F^k y_n(n+1) \cdot e^{-2(n+1)} \geq 1$$

showing that the sets $F^k(U(n, a))$ do not shrink into the set $U(n+1, 1)$ as would follow from the Lemma and the contradiction thus obtained accomplishes the proof of our theorem.

Remark. If X is a metrizable topological space and $F : X \to X$ a continuous selfmapping then the sufficient and necessary conditions for F to be a topological contraction have been found by Ph. Meyers ([3]). It is an open problem to establish a similar characterization for generalized contractions dropping at the same time the hypothesis of metrizability of the space X. The question is:

Given a completely regular topological space X, how to characterize those continuous selfmappings $F : X \to X$ for which there exists a family $\{\rho_i \mid i \in I\}$ of pseudometrics ρ_i on X inducing the topology of X and a constant $q \in (0, 1)$ such that

$$\rho_i(Fx, Fy) \leq q\rho(x, y)$$

for all $x, y \in X$ and all $i \in I$?

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[3] Ph. R. MEYERS, *A converse to Banach's contraction theorem*, «J. Res. Nat. Bur. Standards», ser. B71B, 73–76 (1967).