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On a semigroup approach to oceanography: the treatment of the linearized equations for non viscous sea

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Idrodinamica. — *On a semigroup approach to oceanography: the treatment of the linearized equations for non viscous sea.* Nota di

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RIASSUNTO. — Si studiano le equazioni linearizzate di un noto problema di idrodinamica, riguardante un « mare » non viscoso con densità dipendente dalla profondità. Usufruendo della teoria dei semigruppi, si deriva e discute un'equazione generale inerente a detto problema, e si analizza una soluzione che può essere descritta mediante funzioni elementari.

1. INTRODUCTION

We want to discuss in this Note the equations which describe the motion of an inhomogeneous rotating fluid. For sake of simplicity we neglect the viscosity.

The importance of these problems can be recognized in the effects of a variation of the atmospheric pressure on the currents, in the variation of the sea level near the shore, in the effects of the storms and in the importance of the salinity and heating in the currents. These arguments have interested many researchers. Eckman [1] started in 1905 from the linearized Navier-Stokes equations. He used rather complex and direct calculations. The same problems have been analyzed by Nomitsu [1] by using the Fourier series expansion. For the same problem Crease [1] used the Green functions and Lauweirier [1] and Crepon [1] the Laplace transform techniques. More recently Hansen [1] applied the computer possibilities to these problems. In this situation we feel that the semigroup theory—which is a very general tool—can be fruitfully applied to these problems.

In order to describe our idea, we must say that in this paper we are more interested in the rigorous developments of the theory than in its phenomenological applications. In practice we study the linear version of the equations of the motion for a non viscous fluid, initially stratified and at rest [1]. This is the case of a sea of variable density $\rho(x, y, z, t)$, developing from a situation in which the density is a function of the vertical coordinate only $\rho|_{t=0} = \rho_0(z)$. In the hypothesis that our systems is at $t = 0$ in a condition of stable equilibrium, and that the external perturbations can produce only small changes, our equations are in the first order approximation [2]:

$$
\begin{aligned}
&\rho_0\left(\frac{\partial}{\partial t}u + \mu w - \lambda v\right) = -\frac{\partial}{\partial x}p + F^{(x)}\\
&\rho_0\left(\frac{\partial}{\partial t}v + \lambda u\right) = -\frac{\partial}{\partial y}p + F^{(y)}\\
&\rho_0\left(\frac{\partial}{\partial t}w - \mu u + \frac{g}{\rho_0}\delta\right) = -\frac{\partial}{\partial z}p + F^{(z)}\\
&\frac{\partial}{\partial t}\delta + \frac{d\rho_0}{dz}\cdot w = 0\,, \qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0\,.
\end{aligned}
\tag{1}
$$

(*) Nella seduta del 9 febbraio 1974.

The meaning of the symbols adopted above is as follows:

$\bar{u} = (u, v, w)$ is the eulerian velocity perturbation of first order;

$p(x, y, z, t)$ is the pressure perturbation from initial pressure distribution; we will refer to it as to the pressure;

$\delta = \rho(x, y, z, t) - \rho_0$ is the instantaneous variation of the initial density;

λ, μ are the parameters of the vertical and horizontal Coriolis acceleration respectively. If the x-axis has the direction of the parallel and the y-axis that of the meridian, and the z-axis is directed upwards, $\Omega = \frac{1}{2}(0, \mu, \lambda)$ is the angular velocity of the Earth. If θ is the latitude angle, we have:

$$\lambda = 2\Omega \sin \theta \quad , \quad \mu = 2\Omega \cos \theta ;$$

$\bar{g} = (0, 0, -g)$ is the gravity force;

$\bar{F} = (F^{(x)}, F^{(y)}, F^{(z)})$ is the total resultant of all other forces.

In what follows we treat the general problem and we obtain the equation for the pressure p in a very simple way, by using simple results of the semigroup theory. Moreover we will put in the present Note $\mu = 0$: the case $\mu \neq 0$ will be discussed in a subsequent paper.

In the next § 2 we shall describe the main theorems and applications of the semigroup theory (in a simplified form). Then, in § 3, we arrive at an integral equation for the pressure p. An elementary treatment in the case of constant depth h and $\rho_0 = Ae^{\alpha z}$ is analyzed in the last § 4.

2. Semigroup theory

We now quote the main results of the theory of semigroup. Further results and all the proofs may be found in the related literature [3].

We start with an intuitive introduction of the argument; then we list some foreseeable results and lastly we give them an exact form by stating the rigorously established theorems of the theory.

Let us first consider the following differential equation in the real variable t for the deal function $f(t)$:

$$\frac{d}{dt} f(t) = af(t) \quad , \quad f(0) = F \qquad (a, F \text{ real numbers}), \tag{2}$$

having the obvious solution

$$f(t) = e^{ta} F. \tag{3}$$

The corresponding inhomogeneous problem

$$\frac{d}{dt} f(t) = af(t) + b(t) \quad , \quad f(0) = F \tag{4}$$

has the solution:

(5) $$f(t) = e^{ta}\,\mathrm{F} + \int_0^t \mathrm{d}s\, e^{(t-s)a}\, b(s)\ ;$$

this is known as the Duhamel principle.

Our purpose is to generalize these results in a Banach space X [9]. This generalization requires a theory able to extend the notion of exponential function to the operator space. In fact, we want to solve in a Banach space X the differential equation:

(2′) $$\frac{\mathrm{d}}{\mathrm{d}t}\, x(t) = \mathrm{A}x(t) \quad , \quad x(0) = x_0 \qquad (x(t) \in \mathrm{X}\ ,\ \forall t \geq 0),$$

where $x(t)$ is an element of X, continuous function of the real non-negative variable t, and A is a linear operator upon X.

The equation (2′) may be solved only if the operator A has some " nice " properties, which are precisely the effective object of the study in the semigroup theory. The formal solution of equation (2′) would be

(3′) $$x(t) = e^{t\mathrm{A}}\, x_0\,,$$

where a possible definition of the operator $e^{t\mathrm{A}}$ is:

(6) $$e^{t\mathrm{A}} \equiv \sum_{0=k}^{\infty} \frac{(t\mathrm{A})^k}{k!}\,.$$

However, *a priori* we are not sure that the series (6) has any meaning. In fact, one could easily construct some examples where $e^{t\mathrm{A}}$ does not make sense at all. The semigroup theory shows that the operator $e^{t\mathrm{A}}$ on the Banach space exists, if at least one of the following three properties holds:

i) the A operator is bounded—in which case $e^{t\mathrm{A}}$ exists and is a continuous function of t, called the semigroup generated by A;

ii) if the Banach space X is a Hilbert space, too, and if A is antihermitean (that is: $\mathrm{A}^+ = -\mathrm{A}$); then, not only $e^{t\mathrm{A}}$ exists but it forms also a one-parameter group;

iii) (Hille-Yosida Theorem) if the A operator has its domain dense in X and is closed on it, and if it possesses the further property that, for two real numbers M and ω, we have:

$$\|(\lambda \mathrm{I} - \mathrm{A})^{-n}\| \leq \mathrm{M}(\lambda - \omega)^{-n} \qquad (\lambda > \omega\ ,\ n = 1\ ,\ 2\ ,\cdots),$$

where $\|\ \|$ denotes the norm in the X-space, I the identity operator, λ a real parameter; then exp $(t\mathrm{A})$ does not exist properly: but we can find a semigroup (see the exact definition in the following), which we call $e^{t\mathrm{A}}$ in a formal way and which yields the solution of our equation according to (iii). The last statement is rather complex to verify, but it is the most useful one in practical cases.

We give now the rigorous mathematical formulation of the above mentioned theorems [3].

Let X be a Banach space, which is a complete linear space with a norm. We study a set of linear bounded operators on X which are labelled by a real non negative parameter *t*. Such a family of operators, to be denoted by $\{U(t)\}_{t\geq 0}$, is said to be a strongly " continuous semigroup " if

i) $U(0) = 1$,

ii) $U(t+s) = U(t)U(s) \qquad (\forall s, t > 0)$,

iii) $U(t)x$ is continuous in t in the strong topology for any x. If $U(t)$ is continuous in t in the uniform operator topology, the semigroup too is called " uniformly continuous ".

These definitions are similar to those of the group theory, but only for nonnegative *t*, since the inverse of an element could not exist in the semigroup.

As in group theory, it is useful to introduce the concept of " infinitesimal generator ", to be denoted by A:

$$A \equiv \lim_{s\to 0} \frac{U(s)x - x}{s}, \qquad \forall x \in D(A),$$

where D(A), the domain of A, is just the set of those $x \in X$ such that the limit exists.

If the semigroup is uniformly continuous, the generator A may be defined directly by means of the following limit:

$$A \equiv \lim_{s\to 0} [U(s) - I]/s.$$

It is possible to show, if the semigroup $U(t)$ is

strongly continuous

a) D(A) is dense in X and A is closed on D(A);

b) if $x \in D(A)$, $U(t)x \in D(A)$, for any $t \geq 0$;

c) $\frac{d}{dt}[U(t)x] = AU(t)x = U(t)Ax$, for every $x \in D(A)$;

d) $U(t)x = \lim_{s\to 0} \exp\left[t\frac{U(s)-I}{s}\right]x$, for every $x \in X$.

uniformly continuous

a') A is a bounded operator with D(A);

c') $\frac{d}{dt}U(t) = AU(t) = U(t)A$;

d') $U(t) = e^{tA} \equiv \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.

The main results which we shall use here and elsewhere are:

THEOREM I. *A semigroup* U(*t*) *possesses an infinitesimal generator* A *bounded on, if and only if it is uniformly continuous.*

THEOREM II. *A closed operator* A, *with domain dense in* X, *is the generator of a strongly continuous semigroup, if and only if there exist two real* *numbers* M *and* ω *such that, for every* $\lambda > \omega$, *the operator* $(\lambda I - A)^{-1}$ *exists and satisfies the following estimates*:

$$\|(\lambda I - A)^{-n}\| \leq M(\lambda - \omega)^{-n} \qquad (n = 1, 2, 3, \cdots).$$

THEOREM III. *If the Banach space is also a Hilbert space and the infinitesimal generator is antihermitean* (*that is* $A + A^{+} = 0$), *the semigroup is extensible to an entire group by letting t be negative.*

THEOREM IV. *If the operators* A *and* B *are the infinitesimal generators of two semigroups and moreover* AB = BA, *then* A + B *is the generator of a semigroup and* $e^{t(A+B)}$ *is given by the product semigroup* $e^{tA}\, e^{tB}$.

THEOREM V. *If* A *is the generator of a semigroup and* B *is a linear bounded operator,* A + B *is the infinitesimal generator of a semigroup* (*even if* A *and* B *do not commute*).

We thus have powerful methods for finding quickly the solution of the following equation in X:

$$\frac{d}{dt}\, x(t) = Ax(t) \quad , \quad x(0) = x_0 \qquad (x_0\,,\, x(t) \in D(A)).$$

The solution is immediate and given by

$$x(t) = e^{tA}\, x_0$$

if A is the infinitesimal generator of a semigroup. In the same way, the corresponding inhomogeneous equation

$$\text{(4')} \qquad \frac{d}{dt}\, x(t) = Ax(t) + B(t) \quad , \quad x(0) = x_0 \qquad (x(t) \in D(A)\,,\, B(t) \in X)$$

has a solution which, as can be easily shown, is obtained by generalizing the Duhamel principle, given by

$$\text{(5')} \qquad x(t) = e^{tA}\, x_0 + \int_0^t ds\, e^{(t-s)A}\, B(s)\,.$$

3. OCEAN WITHOUT CURRENTS

In the present assumption our system of equations becomes

$$\text{(1a)} \qquad \frac{\partial}{\partial t}\, u = \lambda v - \frac{1}{\rho_0}\left(\frac{\partial}{\partial x}\, p + F^{(x)}\right),$$

$$\text{(1b)} \qquad \frac{\partial}{\partial t}\, v = -\lambda u - \frac{1}{\rho_0}\left(\frac{\partial}{\partial y}\, p + F^{(y)}\right),$$

$$\text{(1c)} \qquad \frac{\partial}{\partial t}\, w = -\frac{g}{\rho_0} - \frac{1}{\rho_0}\left(\frac{\partial}{\partial z}\, p + F^{(z)}\right),$$

$$\text{(1d)} \qquad \frac{\partial}{\partial t}\, \delta = -\frac{d\rho_0}{dz}\cdot w\,,$$

$$\text{(1e)} \qquad \frac{\partial}{\partial x}\, u + \frac{\partial}{\partial y}\, v + \frac{\partial}{\partial z}\, w = 0\,.$$

It is easy to recognize that, if we assume that the pressure p is a known function, the first two equations are independent from the others. So are the third and fourth equations. We can then solve these two pairs of equations separately, in order to obtain u, v, w and δ as functions of the given function p. After that, we may use the last equation (1e), i.e., the continuity equation, to determine p and so explicitly all other unknowns.

To simplify the computations we introduce the "velocity"

$$W = \frac{g}{\nu} \frac{\delta}{\delta_0},$$

where ν is a frequency defined by:

$$\nu(z) = \sqrt{-\frac{g}{\rho_0} \frac{d\rho_0}{dz}} \qquad \left(\text{remember that: } \frac{d\rho_0}{dz} \leq 0\right). \tag{2}$$

In terms of the new velocity W, the equations (1a), $\cdots$, (1e) may be rewritten:

$$\frac{\partial u}{\partial t} = \lambda v - \frac{1}{\rho_0}\left(\frac{\partial}{\partial x} p + F^{(x)}\right), \tag{1'a}$$

$$\frac{\partial v}{\partial t} = -\lambda u - \frac{1}{\rho_0}\left(\frac{\partial}{\partial y} p + F^{(y)}\right), \tag{1'b}$$

$$\frac{\partial w}{\partial t} = -\nu W - \frac{1}{\rho_0}\left(\frac{\partial}{\partial z} p + F^{(z)}\right), \tag{1'c}$$

$$\frac{\partial W}{\partial t} = \nu w, \tag{1'd}$$

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial z} w = 0. \tag{1'e}$$

Since the two pairs of equations (1'a) (1'b) and (1'c) (1'd) have clearly the same structure, we may study and solve them at the same time. For example, the first two may be written in the form

$$\frac{d}{dt} x(t) = A x(t) + B(t) \quad , \quad x(0) = x, \tag{4'}$$

with $x(t) = \begin{pmatrix} u \\ v \end{pmatrix}$ vector of a (not yet precised) Banach space, $x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ our ocean having been assumed to be at rest at $t = 0$, and:

$$A = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \quad , \quad B(t) = -\frac{1}{\rho_0} \begin{pmatrix} \frac{\partial}{\partial x} p + F^{(x)} \\ \frac{\partial}{\partial y} p + F^{(y)} \end{pmatrix}; \tag{3}$$

we are therefore led to study the semigroup:

$$e^{tA} = e^{t\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}}.$$

It is easy to see that

$$e^{tA} = \begin{pmatrix} \cos \lambda t & \sin \lambda t \\ -\sin \lambda t & \cos \lambda t \end{pmatrix}, \tag{4}$$

since it satisfies

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A \quad , \quad e^{tA}\big|_{t=0} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The fact that the semigroup exists was foreseeable, by looking at the structure of the operator A; for A is obviously limited, and so it generates a uniformly continuous semigroup, extensible to a group.

Before we continue through, we need to define in what space (in any way, a Banach space) our considerations are to be taken as valid. From a physical view-point, if the ocean is bounded held in space, we may suppose that its kinetic energy is bounded and this would suggest as space X the orthogonal sum of L_2 with itself: $X = \bigoplus_{k=1}^{4} L_2^{(k)}$. But this choice has the mathematical defect that the derivative of a function of L_2 is not necessarily in L_2 and this would complicate our task very much. So, starting on physical ground from the fact that the derivatives of u, v, w have to be fairly regular functions of t, having the meaning of eulerian accelerations, we shall restrict ourselves to those functions of L_2 which have their derivatives of first order in L_2, which is perfectly equivalent to choosing as our space the Sobolev [3] space $H^{1,2}$. All the preceding considerations can be repeated in order to find:

$$e^{t\begin{pmatrix} 0 & -\nu \\ \nu & 0 \end{pmatrix}} = \begin{pmatrix} \cos \nu t & -\sin \nu t \\ \sin \nu t & \cos \nu t \end{pmatrix}. \tag{5}$$

Obviously, even if the Coriolis parameter λ were constant in the space variables, our formulae would have been the same.

The solution of eqs. (1′) now, according to the generalized Duhamel principle (5′), are:

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\frac{1}{\rho_0} \int_0^t ds \begin{pmatrix} \cos \lambda (t-s) & \sin \lambda (t-s) \\ -\sin \lambda (t-s) & \cos \lambda (t-s) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} p + F^{(x)} \\ \frac{\partial}{\partial y} p + F^{(y)} \end{pmatrix}, \tag{6a}$$

$$\begin{pmatrix} w \\ W \end{pmatrix} = -\frac{1}{\rho_0} \int_0^t ds \begin{pmatrix} \cos \nu (t-s) & -\sin \nu (t-s) \\ \sin \nu (t-s) & \cos \nu (t-s) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} p + F^{(z)} \\ 0 \end{pmatrix}. \tag{6b}$$

In this way we have been able to write in most cases the general solution of our mathematical problem, as a function of pressure p. We remark that, up to this point, no approximation has been made at all.

From the explicit form of the u, v, w function we can write the continuity equation as an equation in the pressure p. Let us perform the derivatives of u, v, w in eq. (6) under the integral sign, assuming for simplicity that $\bar{F} = \text{grad}\ \mathfrak{V}$. Then, putting $p' = \mathfrak{V} + p$, we have

$$\frac{\partial u}{\partial t}(x, y, z, t) = -\frac{1}{\rho_0}\int_0^t ds \left\{ \cos\lambda(t-s)\frac{\partial^2}{\partial x^2} p + \sin\lambda(t-s)\frac{\partial^2}{\partial x\,\partial y} p - \right.$$

$$\left. -(t-s)\frac{\partial\lambda}{\partial x}\sin\lambda(t-s)\cdot\frac{\partial p}{\partial x} + (t-s)\frac{\partial\lambda}{\partial x}\cos\lambda(t-s)\cdot\frac{\partial p}{\partial y}\right\}(x, y, z, s),$$

$$\frac{\partial v}{\partial t}(x, y, z, t) = -\frac{1}{\rho_0}\int_0^t ds \left\{ -\sin\lambda(t-s)\frac{\partial^2 p}{\partial x\,\partial y} + \cos\lambda(t-s)\frac{\partial^2 p}{\partial y^2} - \right.$$

$$\left. -\cos\lambda(t-s)\cdot\frac{\partial\lambda}{\partial y}(t-s)\cdot\frac{\partial p}{\partial x} - (t-s)\frac{\partial\lambda}{\partial y}\sin\lambda(t-s)\frac{\partial p}{\partial y}\right\}(x, y, z, s),$$

$$\frac{\partial}{\partial t} w(x, y, z, t) = -\int_0^t ds \frac{\partial}{\partial z}\left\{\frac{1}{\rho_0}\cos\nu(z)(t-s)\frac{\partial p}{\partial z}\right\}(x, y, z, s).$$

From the continuity equation, we then obtain for p the equation

$$\int_0^t ds \left[\left\{\frac{1}{\rho_0}\cos\lambda(t-s)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p + \frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\cos\nu(z)(t-s)\frac{\partial}{\partial z} p\right]\right\} + \right.$$

$$+\left\{(t-s)\sin\lambda(t-s)\left(-\frac{\partial\lambda}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial\lambda}{\partial y}\frac{\partial p}{\partial y}\right) + (t-s)\cos\lambda(t-s)\cdot\right.$$

$$\left.\left.\cdot\left(\frac{\partial\lambda}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial\lambda}{\partial y}\frac{\partial p}{\partial \lambda}\right)\right\}\right](x, y, z, s) = 0.$$

If we now assume λ to be constant, we obtain more treatable equations:

$$\frac{\partial}{\partial t} u = -\frac{1}{\rho_0}\int_0^t ds \left[\cos\lambda(t-s)\frac{\partial^2 p}{\partial x^2} + \sin\lambda(t-s)\frac{\partial^2 p}{\partial x\,\partial y}\right](x, y, z, s),$$

$$\frac{\partial}{\partial t} v = -\frac{1}{\rho_0}\int_0^t ds \left[-\sin\lambda(t-s)\frac{\partial^2 p}{\partial x\,\partial y} + \cos\lambda(t-s)\frac{\partial^2 p}{\partial y^2}\right](x, y, z, s),$$

$$\frac{\partial}{\partial t} w = -\int_0^t ds \frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\cos\left\{\nu(t)(t-s)\frac{\partial p}{\partial z}\right\}\right](x, y, z, s).$$

From the continuity equation we deduce

$$\int_0^t ds \left\{\frac{1}{\rho_0}\cos\lambda(t-s)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p + \right.$$

$$\left. + \frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\cos\nu(z)(t-s)\frac{\partial}{\partial z} p\right]\right\}(x, y, z, s) = 0.$$

This equation is fairly general, if one wants to study a non-viscous sea. The only hypotheses are, as described above, that λ is a constant, the system is at rest at $t = 0$ and $\rho|_{t=0} = \rho_0(z)$. Moreover we confine ourselves to studying small departures from the equilibrium, by using linearized equations of motion. We would like to stress, now, that the effect of using the semigroup approach can be seen in the structure of this equation. In fact, the integral equations obtained for the velocities allow that the final equation for the pressure is a 2nd order differential equation [4]. Different formulations give in many cases differential equations of higher order, which are more difficult to treat and which need many more information about the boundary conditions.

In our case, the boundary conditions can be quickly described. On the rigid boundary one can put $\bar{u} \cdot \bar{n} = 0$ if there is no friction between the rigid boundary and the sea water; otherwise we have $\bar{u} = 0$. In terms of p, these conditions are more complex and can be obtained by using the expressions u, v, w as functions of p, as derived above.

The air-water interface, however, is more complex to describe. This is in fact a moving boundary, which could considerably complicate our problem. We can then schematize on physical ground this " dangerous " aspect. If we suppose all the air system is moving under an external force acting on the sea surface, we can assume that $p(x, y, 0, t)$ is a given function $p_0(x, y, t)$. In the other case, where we study the free oscillations of a basin, we can assume that the water profile is given by $z = \zeta(x, y, t)$. The physical hypothesis is

$$p(x, y, 0, t) = g\rho_0 \xi(x, y, t),$$

that is, a reasonable version of the hydrostatic approximation for the $z = 0$ plane. In this way, because

$$\frac{\partial}{\partial t} \zeta = w \Big|_{z=\xi\sim 0},$$

we have

$$\frac{\partial^2}{\partial t^2} p = -g \frac{\partial p}{\partial z} \Big|_{z=0}.$$

In the present Note, however, we shall not treat the boundary problem in a complete way, but only apply the preceding considerations to a simple example.

4. A SIMPLE CASE

We want to study the explicit solution in the case that $\rho_0(z) = e^{-\alpha z}$. This case is well known in the literature: if the depth is constant, the problem can be easily solved [2], [5].

Let us suppose that the system is at rest at $t = 0$. Then an external force is applied to the air-water interface and the system reacts to this force. In particular, we shall suppose that this external force acts in such a way as to fix the shape ξ of the free air-water surface. We can then calculate the pressure at the $z = 0$ surface by using the relation

$$p = \rho_0 g \xi \big|_{z=0} .$$

In practice, by making the Laplace transform of the equation of motion and of the boundary condition, we obtain

$$\frac{\partial}{\partial z}\left\{e^{\alpha z}\frac{1}{\omega^2+g\alpha}\frac{\partial}{\partial z}\tilde{p}\right\}+e^{\alpha z}\frac{1}{\omega^2+\lambda^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right)\tilde{p}=0 ,$$

$$\tilde{p}(x, y, z=0, \omega)=g\rho_0(0)\,\tilde{\xi}(x, y, \omega),$$

$$\frac{\partial\tilde{p}}{\partial z}(x, y, z=-h, \omega)=0 .$$

The last condition comes from the requirement $w(-h)=0$, by using the semigroup relation between w and $\frac{\partial p}{\partial z}$. We can now simplify our problem further by assuming

$$\tilde{\xi}(x, y, \omega)=\mathcal{A}(\omega)\sin Kx ,$$

where $\mathcal{A}(\omega)$ is an analytic function. It is easy to see that the solution of this system is

$$\tilde{p}(x, y, z, \omega)=\rho_0(0)g\,\mathcal{A}(\omega)\sin kx\, e^{-\frac{\alpha}{2}z}\cosh[\gamma z]\{1+\operatorname{tgh}[\gamma h+\beta]\operatorname{tgh}[\gamma z]\},$$

$$\gamma=\sqrt{\frac{\alpha^2}{4}+k^2\frac{\omega^2+\alpha g}{\omega^2+\lambda^2}} , \qquad \beta=\operatorname{sett\,sinh}\left[\frac{\alpha}{2k}\sqrt{\frac{\omega^2+\lambda^2}{\omega^2+g\alpha}}\right].$$

If now one wants to invert this Laplace transform, one can easily find the zero's of cosh $(h\gamma)$. Then one finds that the poles are the solutions of this equation

$$\operatorname{tgh}[\gamma h]=\frac{\gamma}{-\frac{\alpha}{2}}$$

and these are the frequencies of the natural response of our system to any external effect.

The same problem, as well as the calculation of free modes of this systems, can be found in an article of Saint Guily [5].

We now remark that this solution is more general than it looks. In fact, we could sum various solutions for different boundary conditions. We then have the solution for the fairly general boundary condition

$$\tilde{\xi}(x, y, \omega)=\sum_{k,k',l} D_{kk'l}\operatorname{sen}kx\operatorname{sen}k'y\,\mathcal{A}_l(\omega) ,$$

where obviously $D_{kk'l}$ are real numbers and the $\{\mathcal{A}_l\}$ are a complete set of states.

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