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On the formal structure of the relativistic gravitational theory

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Fisica matematica. — *On the formal structure of the relativistic gravitational theory* (*). Nota di ENZO TONTI, presentata (**) dal Socio B. FINZI.

RIASSUNTO. — La teoria dell'omologia dei complessi di celle nello spazio-tempo conduce ad uno schema di classificazione delle grandezze e delle equazioni della teoria relativistica della gravitazione. Lo schema conduce ad una peculiare decomposizione delle equazioni di campo: le equazioni ottenute hanno la stessa struttura delle equazioni di Maxwell, salvo il diverso carattere tensoriale.

1. INTRODUCTION

It is commonly known that in every physical theory there are physical quantities that are naturally referred to the basic geometrical objects, like points, lines, surfaces, volumes, hypervolumes, etc. The purpose of this paper is to show this connection and to deduce some consequences for the relativistic gravitational theory and for electromagnetism (1).

Let us consider a four-dimensional region of the space-time; in order to evidentiate the basic geometrical entities it is expedient to cover the region by a cell-complex (2). This space-time cell-complex, which we denote by K, exhibits 0-cells, 1-cells, 2-cells, 3-cells and 4-cells, i.e. five geometrical entities. The 0, 1, 2, 3, 4-cells will be denoted by P(point), L(line), S(surface), V(volume) and H(hypervolume) respectively. We shall suppose that the space-time region considered has the structure of a smooth manifold and then we choose as cell-complex K the one formed by a discrete set of three-dimensional coordinate manifolds $x^k =$ constant. Every p-cell of K will be oriented according to the natural orientation of the corresponding p-dimensional coordinate manifold. Since the number of p-dimensional coordinate manifolds passing from a point of $\mathbf{R}^4$ is $\binom{4}{p}$ we can group the p-cells in $\binom{4}{p}$ families. The number of families will be written before the symbol of the p-cell: the resulting notation is the following

$$1\,P \quad 4\,L \quad 6\,S \quad 4\,V \quad 1\,H. \tag{1.1}$$

It will be useful to consider also the dual cell-complex, that we denote $\tilde{K}$, whose 0-cells are the baricentric points of the 4-cells of K. The p-cells of $\tilde{K}$ will be denoted with the corresponding symbol of the cell of K with a tilde

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(1) For the implications of such connection between physics and geometry see the author's paper [1].

(2) The study of cell complexes is the subject of algebraic topology [2], [16].

upon the letter. Since to every p-cell of K there corresponds a $(n-p)$-cell of $\tilde{K}$ we can denote this pairing with the scheme

$$\begin{array}{llllll} K: & 1\,P & 4\,L & 6\,S & 4\,V & 1\,H \\ \tilde{K}: & 1\,\tilde{H} & 4\,\tilde{V} & 6\,\tilde{S} & 4\,\tilde{L} & 1\,\tilde{P}. \end{array} \tag{1.2}$$

In this way we have obtained a rational classification of the basic geometrical entities of the space-time. We shall suppose that the p-cells of K be numbered according to some criterion and that the $(n-p)$-cell of $\tilde{K}$ has the same number of the corresponding p-cell of K. If to every p-cell we put into correspondence an element of an additive abelian group $\mathcal{G}$ we have constructed a *p-chain* with *coefficients in* $\mathcal{G}$ [16, p. 225]. We now propose to show how physical quantities give rise to chains of various orders on K and $\tilde{K}$. Let us consider, for ex., the energy-momentum vector P_a given by [8, p. 112]

$$P_a = \iiint\limits_V T_a{}^{\mu}\, dL_{\mu} \tag{1.3}$$

where $T_a{}^{\mu}$ denotes the stress-energy-momentum pseudo-tensor (3). dL_{μ} denotes the supplementary vector of the 3-vector $dV^{\alpha\beta\gamma}$ i.e.

$$dL_{\mu} = \frac{1}{3!}\, e_{\mu\alpha\beta\gamma}\, dV^{\alpha\beta\gamma} \tag{1.4}$$

$e_{\mu\alpha\beta\gamma}$ is the Levi-Civita covariant tensor-capacity. With every 3-cell $V_{(i)}$ of the cell-complex K we may associate the corresponding vector $P_{a(i)}$: in this way the energy-momentum vector gives rise to a 3-chain whose coefficient group is the linear space $\mathcal{G} = \mathbf{R}^4$. As a second example let us consider the electromagnetic potential A_{α} and the circulation

$$C = \int\limits_l A_{\alpha}\, dL^{\alpha}. \tag{1.5}$$

With every 1-cell $L_{(h)}$ we may associate the corresponding circulation $C_{(h)}$ and then we have defined an 1-chain whose coefficient group is the linear space $\mathcal{G} = \mathbf{R}$. As a third example let us consider the Klein-Gordon wave function ψ for a charged particle: if with every 0-cell $P_{(j)}$ we associate the complex number $\psi(P_{(j)})$ we define a 0-chain whose coefficient group is the linear space $\mathcal{G} = \mathbf{C}$. The main process on a p-chain is that of constructing

(3) The pseudo-tensor nature [5, p. 195] [7, p. 225] of the so called stress-energy-momentum "tensor" is often left in the dark: the reason is that we are not accustomed to consider inversions of the space-time axes. For the pseudo-tensor nature of the corresponding spatial stress "tensor" see [5, p. 195], [3, p. 281]. The first index is written in latin to distinguish it from the remaining ones, written in greek character, for a reason that will be explained at the end of this section. Both kinds of index assume the values 0, 1, 2, 3.

a $(p+1)$-chain by the *coboundary process* [16, p. 299]. This is a two-step process defined as follows: let us consider a p-chain

1) we *transfer* the mathematical entity $g_j \in \mathcal{G}$ referred to the j-th p-cell to its cofaces, i.e. to the $(p+1)$-cells incident in it, with the same or opposite sign according if the orientation of the coface agrees or not with that of the p-cell;

2) for every $(p+1)$-cell we *sum* the mathematical entities associated with it by the operation illustrated in 1).

In this way we obtain a $(p+1)$-chain b_{p+1} that is called the *coboundary* of the given p-chain a_p and we write

$$b_{p+1} = \delta a_p \tag{1.6}$$

the operator δ that describes this process is called the *coboundary operator* [2], [16]. It is a remarkable fact that *many basic physical equations asserts that a given $(p+1)$-chain is the coboundary of another p-chain.* Of this kind are the balance equations, the circuital equations, those that define the gradients of field potentials, compatibility equations and those giving the general solutions of a balance equation in field theories [1]. To transfer these matters in the language of the field theories we must consider infinitesimal p-cells: one can see that a p-chain becomes a *differential exterior form of degree p* with values in the abelian group $\mathcal{G}$, or briefly, a *p-form* with values in $\mathcal{G}$ [12, p. 42]. Moreover it can be shown [1] that the coboundary of a chain becomes the *exterior differential* of the form. So let

$$p_a = \frac{1}{3!} T_{a\alpha\beta\gamma} \, dV^{\alpha\beta\gamma} \tag{1.7}$$

be the infinitesimal amount of energy-momentum associated with every 3-cell of a cell-complex. Since the 3-vector $d\mathbf{V}$ can be written as (4)

$$d\mathbf{V} = d\boldsymbol{u} \wedge d\boldsymbol{v} \wedge d\boldsymbol{w} \tag{1.8}$$

where $d\boldsymbol{u}$, $d\boldsymbol{v}$, $d\boldsymbol{w}$ are three vectors, we can write eq. (1.7) as follows

$$p_a = T_{a\alpha\beta\gamma} \, du^\alpha \, dv^\beta \, dw^\gamma . \tag{1.9}$$

This is just a 3-form with values in $\mathbf{R}^4$. The presentation we have given shows that *the differential form gives the infinitesimal amount of the physical quantity associated with the p-vector formed by the p vectors* $d\boldsymbol{u}$, $d\boldsymbol{v}$, $\cdots$. Using the supplementary 1-vector of $d\mathbf{V}$ given by eq. (1.4) the infinitesimal

(4) We use boldfaces characters for vectors, multivectors and elements of the coefficient space $\mathcal{G}$. The symbol $\wedge$ denotes the exterior product of multivectors: the multivector calculus is the essential tool to pass from algebraic topology to the theory of exterior differential forms [1].

amount p_a can be written

$$p_a = T_b^{\mu}\, dL_{\mu} \,. \tag{1.10}$$

This is the *adjoint* 1-form [5]. We emphasize the fact that the adjoint form gives the same infinitesimal amount of the physical quantity associated with the 3-vector $d\mathbf{V}$. More in general if ω denotes the infinitesimal amount of a physical quantity referred to an infinitesimal p-cell $d\boldsymbol{p}$ we may write

$$\omega = \mathcal{A}\, d\boldsymbol{p} \tag{1.11}$$

where $\mathcal{A}$ denotes a linear operator from the space of p-vectors to the linear space $\mathcal{G}$. Using the components of $d\boldsymbol{p}$ in a local coordinate system we may write

$$\omega = \frac{1}{p!}\, \boldsymbol{a}_{\alpha\beta\cdots}(\boldsymbol{x})\, dp^{\alpha\beta\cdots} \equiv \boldsymbol{a}_{\alpha\beta\cdots}(\boldsymbol{x})\, du^{\alpha}\, dv^{\beta}\cdots \tag{1.12}$$

and moreover using a base in the coefficient space $\mathcal{G}$

$$\omega_{ab\cdots} = \frac{1}{p!}\, a_{ab\cdots\alpha\beta\cdots}(\boldsymbol{x})\, dp^{\alpha\beta\cdots} \equiv a_{ab\cdots\alpha\beta\cdots}(\boldsymbol{x})\, du^{\alpha}\, dv^{\beta}\cdots \,. \tag{1.13}$$

We introduce the name of *passive indices* for the indexes that are relative to the mathematical nature of ω (scalar vector, tensor, spinor, matrix, operator, etc.) and use for them latin characters: $a, b, \cdots$. Moreover we shall call *active indices* those that are relative to the p-vector $d\boldsymbol{p}$ and use for them greek indexes: $\alpha, \beta, \cdots$. The set of functions $a_{ab\cdots\alpha\beta\cdots}(\boldsymbol{x})$ has a tensor behaviour with respect to the active indexes and has an a priori unprecised nature with respect to the passive indices: we say that they form a *tensor-object*. The functions $a_{ab\cdots\alpha\beta\cdots}(\boldsymbol{x})$ is, in a sense, a kind of *density* of the physical quantity ω. On account of the skew-symmetry of the tensor character of $dp^{\alpha\beta\cdots}$ there is not loss of generality to consider the $a_{ab\cdots\alpha\beta\cdots}(\boldsymbol{x})$ as forming a skew-symmetric tensor field with respect to the active indices [3, p. 281].

As an example let us consider tha parallel transport of a vector along an infinitesimal circuit described by a bivector $d\boldsymbol{b}$. The rotation of the vector can be described by an infinitesimal bivector ω: the linear relation between them is represented by the equation

$$\omega_{ab} = \frac{1}{2!}\, R_{ab\alpha\beta}\, db^{\alpha\beta} \,. \tag{1.14}$$

The operator $R_{ab\alpha\beta}$ is the Riemann tensor. This clearly exhibits that the Riemann tensor has two passive indices and two active indices. Using the

(5) Also called *conjugated* or *polar* or *starred* form and denoted by a star before the symbol: so if γ denotes a form, $*\gamma$ denotes its dual [13, p. 68].

supplementary $(n-p)$-vector whith components [6]

$$\mathrm{d}q_{\mu\nu\ldots} = \frac{1}{p!}\, e_{\mu\nu\ldots\alpha\beta\ldots}\, \mathrm{d}p^{\alpha\beta\ldots} \tag{1.15}$$

we can write

$$\omega_{ab\ldots} = \frac{1}{(n-p)!}\, \mathrm{A}_{ab\ldots}{}^{\mu\nu\ldots}(\boldsymbol{x})\, \mathrm{d}q_{\mu\nu\ldots} \equiv \mathrm{A}_{ab\ldots}{}^{\mu\nu\ldots}(\boldsymbol{x})\, \mathrm{d}r_\mu\, \mathrm{d}s_\nu \cdots \tag{1.16}$$

where $\mathrm{d}\boldsymbol{q} = \mathrm{d}\boldsymbol{r} \wedge \mathrm{d}\boldsymbol{s} \wedge \cdots$.

If we use the criterion to refer the density field $a_{ab\ldots\alpha\beta\ldots}(\boldsymbol{x})$ or its dual $\mathrm{A}_{ab\ldots}{}^{\mu\nu\ldots}(\boldsymbol{x})$ to the same geometrical element (P, L, S, V, H) to which the corresponding physical quantity ω is referred, then we may construct a classification scheme for the variables and the equations of a field theory (Table I).

2. Relativistic gravitation theory (linear approximation)

We propose to show that the scheme given in Table I classify at the same time the variables and the equations of the relativistic gravitation theory and of the electromagnetic field. The only difference lies in the kind of space of coefficients $\mathcal{G}$ we choose: more precisely

1) electromagnetism: $\mathcal{G} = \mathbf{R}$;
2) relativ. grav. theory: $\mathcal{G} = \mathbf{R}^4$.

In terms of indices the relativistic gravitational theory differs from the electromagnetism for the presence of a passive index. Let $g_{\alpha\beta}$ be the riemannian metric tensor and $\mathring{g}_{\alpha\beta}$ be the metric tensor of a flat space referred to the same set of coordinates. Putting as usual [4, p. 315]

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + h_{\alpha\beta} \tag{2.1}$$

and

$$\varphi_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\, h\mathring{g}_{\alpha\beta} \tag{2.2}$$

we obtain

$$h_{\alpha\beta} = \varphi_{\alpha\beta} - \frac{1}{2}\, \varphi\mathring{g}_{\alpha\beta}\,. \tag{2.3}$$

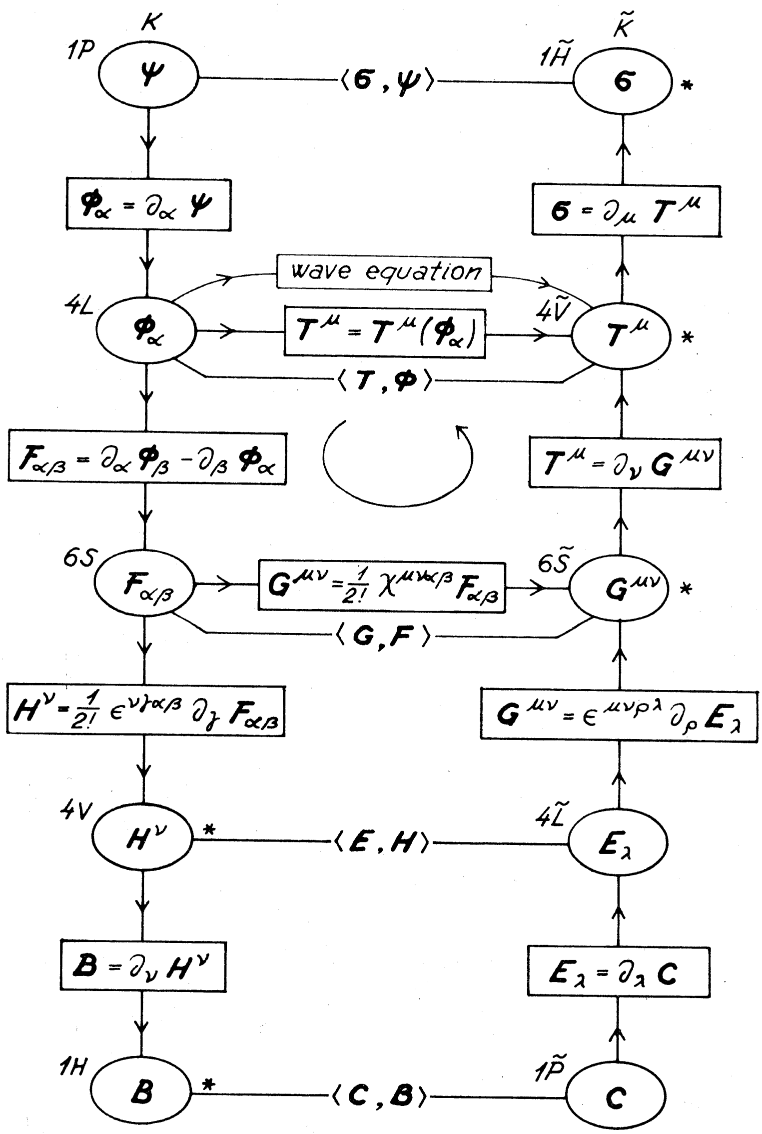
Now we can consider $\varphi_{\alpha\beta}$ as the symmetric part of a second rank tensor $\Phi_{\alpha\beta}$ i.e.

$$\varphi_{\alpha\beta} = \frac{1}{2}\,(\Phi_{\alpha\beta} + \Phi_{\beta\alpha})\,. \tag{2.4}$$

(6) The supplementary of a multivector must be made which the Levi–Civita tensorial capacity $e_{\alpha\beta\ldots}$ or with the corresponding tensorial density $\varepsilon^{\mu\nu\ldots}$ and not with the Ricci tensor $\varepsilon_{\alpha\beta\ldots} = \sqrt{|g|}\, e_{\alpha\beta\ldots}$ or $\varepsilon^{\mu\nu\ldots} = \sqrt{|g|}\, \varepsilon^{\mu\nu\ldots}$. This is needed to maintain the nonmetrical nature to the notion of supplementary of a multivector.

TABLE I

A classification scheme for electromagnetism and relativistic gravitation theory



The star denotes controvariant tensor densities. The number of distinct components of every tensor-object is equal to the number of families of the corresponding coordinate manifolds.

In this way we are led to consider a space-time field whose field potential is a nonsymmetrical second rank tensor $\Phi_{\alpha\beta}$. This amounts to a generalization of the linear gravitational theory that is useful for the case in which the stress-energy-momentum " tensor " is nonsymmetric [15]. This does not mean that the skew-symmetrical part of $\Phi_{\alpha\beta}$ be *alone* sufficient to describe the dynamical features connected with the nonsymmetrical character of the "tensor" $T_a^{\ \nu}$. We are here faced with a situation analogous to the classical continuum mechanics that is described by a symmetric strain tensor while the more realistic notion of *oriented continuum* (also called polar or Cosserat continuum) need a nonsymmetrical strain tensor and, moreover, another so called curvature-twist tensor that is a third-rank tensor [14].

The field potential $\Phi^a_{\ \alpha}$ will be linked to the (nonsymmetrical) pseudo-tensor $T_b^{\ \mu}$ by means of the two equations

$$F^a_{\ \alpha\beta} = \overset{\circ}{\nabla}_\alpha \Phi^a_{\ \beta} - \overset{\circ}{\nabla}_\beta \Phi^a_{\ \alpha} \qquad \overset{\circ}{\nabla}_\nu G_b^{\ \mu\nu} = T_b^{\ \mu} \tag{2.5}$$

and of the constitutive equation

$$G_b^{\ \mu\nu} = \frac{1}{2!} \chi_{b\ \ a}^{\ \mu\nu\ \alpha\beta} F^a_{\ \alpha\beta} \tag{2.6}$$

with [7]

$$\chi_{b\ \ a}^{\ \mu\nu\ \alpha\beta} = -\frac{c^4}{16\pi G} \sqrt{|\overset{\circ}{g}|}\, \overset{\circ}{g}_{ba} (\overset{\circ}{g}^{\mu\alpha} \overset{\circ}{g}^{\nu\beta} - \overset{\circ}{g}^{\mu\beta} \overset{\circ}{g}^{\nu\alpha}) \tag{2.7}$$

that are similar to the equations of the electromagnetic field (see Table II). The direct link between $\Phi^a_{\ \alpha}$ and $T_b^{\ \mu}$ is obtained inserting one equation in the other according to the order indicated by the arrows of Table I. One obtain

$$T_b^{\ \mu} = \overset{\circ}{\nabla}_\nu \left[\frac{1}{2!} \chi_{b\ \ a}^{\ \mu\nu\ \alpha\beta} (\overset{\circ}{\nabla}_\alpha \Phi^a_{\ \beta} - \overset{\circ}{\nabla}_\beta \Phi^a_{\ \alpha}) \right]. \tag{2.8}$$

If we introduce the metrical [5, p. 136] gauge condition

$$\overset{\circ}{\nabla}{}^\alpha \Phi^a_{\ \alpha} = 0 \tag{2.9}$$

the wave equation (2.8) becomes

$$T_b^{\ \mu} = -\sqrt{|\overset{\circ}{g}|} \frac{c^4}{16\pi G} \overset{\circ}{\Box} \Phi_b^{\ \mu}. \tag{2.10}$$

If we take the symmetrical part of the tensors of both members putting

$$\tau_b^{\ \mu} \overset{\text{def}}{=} \frac{1}{2} (T_b^{\ \mu} + T^\mu_{\ b}) \tag{2.11}$$

we obtain

$$\tau_b^{\ \mu} = -\sqrt{|\overset{\circ}{g}|} \frac{c^4}{16\pi G} \overset{\circ}{\Box} \Phi_b^{\ \mu} \tag{2.12}$$

(7) The corresponding constitutive tensor for the electromagnetism is [5, p. 179] (using the metric tensor $\overset{\circ}{g}_{\alpha\beta}$)

$$\chi^{\mu\nu\ \alpha\beta} = \frac{1}{2\mu_0} \sqrt{|\overset{\circ}{g}|} (\overset{\circ}{g}^{\mu\alpha} \overset{\circ}{g}^{\nu\beta} - \overset{\circ}{g}^{\mu\beta} \overset{\circ}{g}^{\nu\alpha}).$$

that is the wave equation of the relativistic gravitational theory for weak fields. We remark that the gauge condition (2.9) is slightly different from Hilbert condition [4, p. 314] that asserts the vanishing of the divergence of the *symmetric* part of $\Phi^a_{\ \alpha}$.

An important property of the formal differential operators of the left and right columns of Table I is the following: let us define the bilinear functionals

$$
\begin{aligned}
&\langle \sigma , \psi \rangle \overset{\text{def}}{=} \iiiint_\Omega \sigma \cdot \psi \,\mathrm{d}\Omega && \langle \mathbf{E} , \mathbf{H} \rangle \overset{\text{def}}{=} \iiiint_\Omega \mathbf{E}_\nu \cdot \mathbf{H}^\nu \,\mathrm{d}\Omega \\
&\langle \mathbf{T} , \boldsymbol{\Phi} \rangle \overset{\text{def}}{=} \iiiint_\Omega \mathbf{T}^\mu \cdot \boldsymbol{\Phi}_\mu \,\mathrm{d}\Omega && \langle \mathbf{C} , \mathbf{B} \rangle \overset{\text{def}}{=} \iiiint_\Omega \mathbf{C} \cdot \mathbf{B} \,\mathrm{d}\Omega \\
&\langle \mathbf{G} , \mathbf{F} \rangle \overset{\text{def}}{=} \iiiint_\Omega \frac{1}{2!} \mathbf{G}^{\mu\nu} \cdot \mathbf{F}_{\mu\nu} \,\mathrm{d}\Omega
\end{aligned}
\tag{2.13}
$$

where the dot denotes tensor composition with respect to the passive indices; $\mathrm{d}\Omega = \sqrt{|\overset{\circ}{g}|}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$. Then we show that *the differential operators of the left and right columns are either formally adjoint or skew-adjoint one of the other.* So

$$
\iiiint_\Omega \mathbf{T}^\mu \cdot (\partial_\mu \Psi) \,\mathrm{d}\Omega \equiv \iiiint_\Omega (-\partial_\mu \mathbf{T}^\mu) \cdot \Psi \,\mathrm{d}\Omega + \{\text{b.t.}\}
\tag{2.14}
$$

where {b.t.} denotes the boundary terms. Moreover

$$
\begin{aligned}
&\iiiint_\Omega \frac{1}{2!} \mathbf{G}^{\mu\nu} \cdot (\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu) \,\mathrm{d}\Omega = \iiiint_\Omega (\partial_\nu \mathbf{G}^{\mu\nu}) \cdot \Phi_\mu \mathrm{d}\Omega + \{\text{b.t.}\} \\
&\iiiint_\Omega \mathbf{E}_\nu \cdot \left(\frac{1}{2!}\, \varepsilon^{\nu\gamma\alpha\beta}\, \partial_\gamma \mathbf{F}_{\alpha\beta}\right) \mathrm{d}\Omega \equiv \iiiint_\Omega \frac{1}{2!} (\varepsilon^{\alpha\beta\gamma\nu}\, \partial_\gamma \mathbf{E}_\nu) \cdot \mathbf{F}_{\alpha\beta} \,\mathrm{d}\Omega + \{\text{b.t.}\}
\end{aligned}
\tag{2.15}
$$

and so on. This property together with the symmetry of the constitutive pseudo-tensor

$$
\chi_a{}^{\alpha\beta}{}_b{}^{\mu\nu} = + \chi_b{}^{\mu\nu}{}_a{}^{\alpha\beta}
\tag{2.16}
$$

permits us to give a rational derivation of the variational formulation of the wave equation of a field theory that enters the scheme of Table I [6]. If we apply the systematic procedure explained in [6] we find the action

$$
\begin{aligned}
\mathrm{S}[\boldsymbol{\Phi}] = &\iiiint_\Omega \frac{1}{2!} \mathbf{G}^{\mu\nu} [\mathbf{F}_{\alpha\beta}(\boldsymbol{\Phi}_\gamma)] \cdot \mathbf{F}_{\mu\nu}(\boldsymbol{\Phi}_\gamma) \,\mathrm{d}\Omega \,- \\
&- \iiiint_\Omega \left[\int_{\lambda=0}^{\lambda=1} \mathbf{T}^\mu (\boldsymbol{\Phi}_\gamma(\lambda)) \cdot \frac{\mathrm{d}\boldsymbol{\Phi}_\mu(\lambda)}{\mathrm{d}\lambda} \,\mathrm{d}\lambda\right] \mathrm{d}\Omega \,.
\end{aligned}
\tag{2.17}
$$

The reason for the integration on the parameter λ in the second integral is given in [9]. As an example for a free field ($T_b^{\mu} = 0$) we obtain the action

$$S_f[\Phi] = \iiiint_{\Omega} \gamma \sqrt{|g|}\, [\mathring{\nabla}^{\mu} \Phi^{\nu} \cdot \mathring{\nabla}^{\mu} \Phi_{\nu} - \mathring{\nabla}^{\nu} \Phi^{\mu} \cdot \mathring{\nabla}_{\mu} \Phi_{\nu}]\, d\Omega \tag{2.18}$$

that is valid for the electromagnetism putting $\gamma = \frac{1}{\mu_0}$ and for the relativistic gravitational theory putting $\gamma = -\frac{c^4}{16\pi G}$.

Summarizing we have shown that the natural association of the physical quantities of the relativistic gravitational field, as well as those of the electromagnetic field, with the basic geometrical entities of space-time leads to a classification scheme of the field tensors and of their connecting equations. Such a classification permits a systematic deduction of many mathematical properties [1].

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