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On the tensor with given divergence on Riemannian manifolds

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Fisica matematica. — *On the tensor with given divergence on Riemannian manifolds* (*). Nota di CARLO MOROSI, presentata (**) dal Socio B. FINZI.

RIASSUNTO. — Si fornisce la soluzione del problema dell'equilibrio, considerato come problema aggiunto della congruenza. A tal fine si utilizzano le condizioni di congruenza ottenute in un precedente lavoro [1], precisando l'eliminazione delle funzioni arbitrarie ivi contenute.

1. INTRODUCTION

The necessary and sufficient congruence conditions, that is the compatibility conditions of the system

$$\xi_{ik} - (s_{i/k} + s_{k/i}) = 0 \qquad (i, k = 1, 2, \cdots, N) \tag{1.1}$$

where ξ_{ik} and s_i are the infinitesimal strain and displacement fields respectively, have been obtained both for the case of Riemannian N-manifolds with a group of rigid motions [1], and for the case of no rigid motions [2].

If there are no rigid motions, the congruence conditions are obtained [2] in an invariant tensor form, by making zero some linear and homogeneous combinations of the strain and its tensor derivatives.

Instead, in the case analyzed in paper [1] they may contain some arbitrary functions; the elimination of these arbitrary functions allows us to give the congruence conditions a linear and homogeneous form: hence we can obtain the general solutions of the equations

$$p^{ki}{}_{/k} = 0 \quad ; \quad p^{ki}{}_{/k} = f^i \qquad (i, k = 1, 2, \cdots, N) \tag{1.2}$$

that we call equilibrium equations, generalizing from the case $N = 2, 3$.

In fact, as suggested by [3], [4] and applied by [2] and [5] to the cases of Riemannian N-manifolds with no rigid motions and of the rotation surface respectively, if the congruence conditions are given an operator form

$$\Omega\xi = 0, \tag{1.3}$$

Ω being the linear congruence operator, and if the equilibrium problem is the adjoint of the congruence problem (1.1), then the general solution of the homogeneous equation (1.2) is

$$p = \tilde{\Omega}\lambda; \tag{1.4}$$

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(**) Nella seduta del 9 febbraio 1974.

therefore it is constructed by means of the adjoint operator $\tilde{\Omega}$ and it depends upon as many arbitrary functions λ as the essential congruence conditions are.

By means of the the congruence conditions in the form (1.3), in this paper we obtain the solutions of Eqs. (1.2), discussing their structure and giving a few simple examples (sec. 2). In order to clarify how to give the congruence conditions the form (1.3), the elimination of the arbitrary functions is analyzed; with the exception of some particular cases, the explicit use of a coordinate system is required, hence the possibility of a directly invariant formulation is lost: the procedure followed and the results obtained are summarized for a few particular cases (sec. 3).

2. THE EQUILIBRIUM PROBLEM

In this section the solutions of Eqs. (1.2), both homogeneous and complete, are given.

a) *General solution of the homogeneous equation.*

We suppose that we can choose boundary conditions physically meaningful so that the equilibrium problem be the very adjoint problem of the congruence.

Thus we can rewrite Eqs. (1.1) and (1.2) in operator form

$$\mathrm{D}s = \xi \tag{2.1}$$

$$\tilde{\mathrm{D}}p = 0 \quad ; \quad \tilde{\mathrm{D}}p = -2f. \tag{2.2}$$

If the congruence conditions (1.3) are given, that is if the congruent strain ξ belongs to the null manifold $\mathfrak{N}(\Omega)$ of the congruence operator Ω, then the solution p of the adjoint problem (that is the solution of the homogeneous Eq. (2.2) to which suitable boundary conditions are associated) belongs to the range $\mathfrak{R}(\tilde{\Omega})$ of the adjoint operator $\tilde{\Omega}$ of Ω [7, 6], hence it has the form (1.4). It can be given another interesting form, making its structure more evident: that will be shown in the following, after discussing the determination of particular solutions of the complete Eq. (2.2).

b) *Particular solutions of the complete equation.*

As for the complete Eq. (2.2), it has a solution if and only if

$$\langle f, v \rangle \equiv \int_\tau f^i v_i \,\mathrm{d}\tau = 0 \tag{2.3}$$

where v_i is the general solution of the Killing equation

$$\mathrm{D}v \equiv v_{i/k} + v_{k/i} = 0\,. \tag{2.4}$$

Now we analyze a few particular cases; firstly we consider the Euclidean space E_3: referring to orthogonal Cartesian coordinates x^k, the rigid displacement field $v_i^{(E)}$ is

$$v_i^{(E)} \equiv a_i + \varepsilon_{ijk}\, x^k\, \omega \tag{2.5}$$

where a_i and ω^j are arbitrary constant parameters and ε_{ijk} is the permutation symbol; then Eq. (2.3) becomes

$$\langle f, v \rangle \equiv a_i\, R^i + \omega^i M_i = 0\,, \tag{2.6}$$

being

$$R^i \equiv \int_\tau f^i\, d\tau \quad ; \quad M_i \equiv \int_\tau \varepsilon_{ijk}\, x^j f^k\, d\tau \tag{2.7}$$

the sum of the field forces and the sum of the moments, about any point, of the field forces; therefore, on account of the arbitrariness of a_i and ω^i, the well-known global equilibrium conditions follow from (2.6).

Now we consider a Riemannian manifold V_N: as for its structure, we recall (see [1] for more details) that the submanifold V_r is given at any $P \in V_N$ by the r Killing congruences and V_{N-r} is its orthogonal complement

$$V_N = V_r \oplus V_{N-r} \tag{2.7}$$

spanned by $(N - r)$ vectors $\underset{\alpha}{A}_{/i}$ ($\underset{\alpha}{A}$ being $(N - r)$ functionally independent invariants constructed by the metric tensor of V_N and its ordinary derivatives).

If $f \in V_{N-r}$, i.e. if we can define

$$f_i \equiv f_\alpha\, \underset{\alpha}{A}_{/i} \qquad (\alpha = r + 1, \cdots, N) \tag{2.8}$$

the condition (2.3) is verified; in particular this is always true if there are no rigid motions ($r \equiv 0$), e.g. for a membrane upon a generic surface. Instead in the case of a rotation surface, a force field is balanced only if it is parallel to the curvature gradient

$$f_i = \lambda K_{/i} \tag{2.9}$$

but not, generally, if it has the form

$$f_i = \lambda K_{/i} + \mu v_i \tag{2.10}$$

(where v_i, parallel to the congruences $K =$ cost., is the Killing vector giving the rigid displacement of the surface), unless μ satisfies to the global condition on Σ: $\int_\Sigma \mu v^2\, d\Sigma = 0$, according to (2.3). Moreover, if f^i has the form (2.8), a particular solution of the complete Eq. (2.2) can be obtained directly in tensor form by only applying differential operators to f_i: in fact as the vector component $u \in V_{N-r}$ of the displacement s solution of Eq. (2.1) has the operator form

$$u = R\xi \tag{2.11}$$

it follows from (2.8)

$$\langle f, s \rangle = \langle f, u \rangle. \tag{2.12}$$

Thus the tensor

$$q_1 \equiv -2\,\tilde{\mathrm{R}} f \tag{2.13}$$

is a particular solution of the complete Eq. (2.2); in fact by (2.13), (2.1), (2.11), (2.12) we have

$$\begin{aligned}\langle \tilde{\mathrm{D}} q_1 + 2f, s \rangle &= -2\langle \tilde{\mathrm{D}}\tilde{\mathrm{R}} f, s \rangle + 2\langle f, s \rangle = -2\langle \tilde{\mathrm{R}} f, \xi \rangle + \\ &+ 2\langle f, s \rangle = -2\langle f, \mathrm{R}\xi \rangle + 2\langle f, u \rangle = 0\end{aligned} \tag{2.14}$$

hence $\tilde{\mathrm{D}} q_1 = -2f$, on account of the arbitrariness of s.

Instead if f satisfies to (2.3) but not to (2.8), it is not possible, we believe, to give a particular solution of the complete Eq. (2.2) only by using *differential* operators on f: in this case [1, sec. 3] the displacement field induced by ξ has the form

$$s = \mathrm{L}\xi + v \equiv \mathrm{R}\xi + \mathrm{S}\xi + v, \tag{2.15}$$

S being a linear *integro-differential* operator; then one can verify, as well as for (2.13), that the tensor

$$q_2 \equiv -2\,\tilde{\mathrm{L}} f = -2\,\tilde{\mathrm{R}} f - 2\,\tilde{\mathrm{S}} f \tag{2.16}$$

is a particular solution of the complete Eq. (2.2).

c) *Structure of the general solution of the complete equation.*

On account of the previous hypotheses and the supposed linearity of the equilibrium problem, the general solution of the complete Eq. (2.2) is given by the tensor

$$p = \tilde{\Omega}\lambda - 2\,\tilde{\mathrm{L}} f. \tag{2.17}$$

To explain the structure of this tensor, we briefly recall how the operator $\tilde{\Omega}$ can be obtained (see [1] for more details).

The necessary congruence conditions (they are easily shown to be also sufficient) have been searched as follows: for any displacement vector field s, solution of Eq. (2.1), the existence conditions of its *intrinsic* components $\underset{\alpha}{s}$ and $\underset{m}{s}$ are considered; the scalar functions $\underset{\alpha}{s}$ are univocally obtained by the strain ξ, being the following linear and homogeneous combinations

$$\underset{\alpha}{s} \equiv \underset{\alpha\beta}{\mathrm{B}}\, \delta\underset{\beta}{\mathrm{A}}(\xi) \qquad (\alpha, \beta = r+1, \cdots, \mathrm{N}) \tag{2.18}$$

of $\delta\underset{\beta}{\mathrm{A}}(\xi)$ [1, sec. 3], whereas the components $\underset{m}{s}$ are the *eventual* solutions of the system

$$\underset{m}{s}_{/i} = \underset{m\alpha}{\Lambda}\, \underset{\alpha}{\mathrm{A}}_{/i} + \underset{ml}{\mathrm{M}}\, \underset{l}{v}_i = \underset{m\alpha}{\Lambda}\, \underset{\alpha}{\mathrm{A}}_{/i} + \underset{ml}{\mathrm{S}}\, \underset{l}{v}_i + \underset{ml}{\mathrm{E}}\, \underset{l}{v}_i \tag{2.19}$$

where $\underset{m\alpha}{\Lambda}$ and $\underset{ml}{S} \equiv (\underset{ml}{M} + \underset{lm}{M})/2 \equiv \underset{lm}{S}$ are *known* scalar functions of the strain, and $\underset{ml}{E} \equiv (\underset{ml}{M} - \underset{lm}{M})/2 \equiv -\underset{lm}{E}$ are arbitrary scalar functions [1, sec. 3].

Two sets of congruence conditions are thus obtained: if the first set is verified, that is if

$$\underset{m}{C}_{ik} \equiv (\underset{m\alpha}{\Lambda}\underset{\alpha}{A}_{/i} + \underset{ml}{M}\underset{l}{v}_i)_{/k} - (\underset{m\alpha}{\Lambda}\underset{\alpha}{A}_{/k} + \underset{ml}{M}\underset{l}{v}_k)_{/i} = 0\,, \tag{2.20}$$

the right hand members of (2.19) are "irrotational", hence the components $\underset{m}{s}$ exist. The second set is obtained by making zero $(N - r)(N - r + 1)/2$ intrinsic components $\underset{\alpha\beta}{\sigma}\ (\alpha\,,\beta = r + 1\,,\cdots, N)$ (which belong to the sub-manifold V_{N-r}) of the tensor

$$\sigma_{ik} \equiv \xi_{ik} - (u_{i/k} + u_{k/i})\,, \tag{2.21}$$

the vector

$$u_i \equiv \underset{\alpha}{s}\underset{\alpha}{A}_{/i} \tag{2.22}$$

being the vector component of s_i on V_{N-r} [1, sec. 3].

The components $\underset{\alpha\beta}{\sigma}$ (containing no arbitrary functions) are linear and homogeneous functions of the strain and its tensor derivatives (they do not exist if $r = N$, i.e. if any displacement field is a linear combination of rigid displacements); furthermore, the arbitrary functions $\underset{ml}{E}$, if they are present, can be eliminated from (2.20) (as will be explained by a few examples in the next section). Therefore the conditions (2.20), as well as the conditions $\underset{\alpha\beta}{\sigma} = 0$, can be given respectively the following operator form

$$\Omega_1\,\xi = 0 \tag{2.23}$$

$$\Omega_2\,\xi = 0 \tag{2.24}$$

corresponding, all together, to (1.3). In contrast with the case $r = 0$ [2], where the highest order of derivatives can be directly determined, in this case it can be obtained only after the elimination of the arbitrary functions.

Furthermore one can easily verify that, if ξ satisfies to (2.20) (i.e. to (2.23)), the conditions (2.24) are equivalent to the following ones

$$(I - DL)\,\xi = 0 \tag{2.25}$$

where I is the identity operator and the operator L is defined by (2.15) and (2.19), which are integrable on account of (2.20).

So the tensor (2.17) can be written as

$$p \equiv p_1 + p_2 - 2\tilde{L}f \tag{2.26}$$

where

$$p_1 \equiv \tilde{\Omega}_1\,\lambda \tag{2.27}$$

$$p_2 \equiv \tilde{\Omega}_2\,\mu = (I - \tilde{L}\tilde{D})\,\mu = \mu - \tilde{L}\tilde{D}\mu\,. \tag{2.28}$$

At last it must be remarked that p_1 exists only if $r \neq 0$ (that is for manifolds with groups of motions), whereas p_2, which is obtained by adding to an arbitrary tensor another tensor constructed by means of $\tilde{L}$ and its divergence, has the same formal structure as the general solution of the homogeneous Eq. (2.2), when there are no groups of motions [2].

In the particular case of a rotation surface (see [5] for more details) we have two congruence conditions, the first one of the form (2.23) and the second one of the form (2.24): hence the solution (2.26), where λ and μ are two arbitrary scalar functions, depends upon these two functions and their tensor derivatives.

3. ELIMINATION OF THE ARBITRARY FUNCTIONS

In the previous section the solution of the homogeneous Eq. (2.2) has been written as the sum of the tensor (2.27) (typical of the manifolds with rigid motions) and of a tensor with the same formal structure as the general solution of Eq. (2.2) for the manifolds with no rigid motions [2]. It must be stressed that one has to calculate the adjoint operator $\tilde{\Omega}_1$ of Ω_1 in order to obtain the tensor (2.27), i.e. the arbitrary functions $\underset{ml}{E}$ must be eliminated from (2.20) (if they are present): in the general case these functions are $r(r-1)/2$, hence they do not exist when $r = 1$ (e.g. on the rotation surface), and they reduce to only one function when $r = 2$ (e.g. on the surface with zero or constant Gaussian curvature). It must be pointed out that the elimination of $\underset{ml}{E}$ does not seem possible if the form of the metric tensor is not specified (this may be not true in some cases, e.g. if $N = 5$ and $r = 3$ the three functions $\underset{ml}{E}$ can be eliminated from (2.20) by solving only an algebraic system): therefore if the congruence conditions are written in a linear and homogeneous form, the invariant formulation is lost, differently from the case analyzed in [2]. In this paper we limit ourselves to showing explicitly the procedure in the simplest case, that of the Euclidean plane, and to summarizing the results for the Euclidean space and the Schwarzschild four-manifold.

For the Euclidean plane ($r = N = 2$), it is $\underset{m\alpha}{\Lambda} = 0$, $\underset{ml}{S} = \frac{1}{2}\,\xi_{ml}$, $\sigma_{ik} = \xi_{ik}$: the congruence conditions corresponding to $\underset{\alpha\beta}{\sigma} = 0$ do not exist ($r = N$) and the conditions (2.20) correspond to the following ones

$$(3.1) \qquad -\frac{1}{2}\,\underset{srm}{Q} = \underset{sl}{E}\,\underset{l}{v}_{i|k}\,\underset{m}{v}^k\,\underset{r}{v}^i + \frac{1}{2}\,\underset{sl}{E}_{|i}\,(\underset{m}{v}^i\,\underset{lr}{V} - \underset{r}{v}^i\,\underset{lm}{V})\,,$$

where $\underset{srm}{Q}$ are known functions of the strain and $\underset{lr}{V} \equiv \underset{l}{v}_i\,\underset{r}{v}^i$.

Referring to orthogonal Cartesian coordinates, the two translations can be chosen as linearly independent Killing vectors: $\underset{l}{v}_i = \delta_{li}$; defining

$$(3.2) \qquad \underset{ml}{E} \equiv A\,(\delta_{m1}\,\delta_{l2} - \delta_{m2}\,\delta_{l1})\,,$$

where the function A is undetermined, the two non-homogeneous congruence conditions are obtained

$$\underset{srm}{\mathrm{Q}} = \mathrm{A}_{/r}(\delta_{s1}\,\delta_{m2} - \delta_{m1}\,\delta_{s2}) + \mathrm{A}_{/m}(\delta_{r1}\,\delta_{s2} - \delta_{s1}\,\delta_{r2}) \tag{3.3}$$

which are equivalent to the unique condition (homogeneous in ξ_{ik})

$$2\,(\underset{112}{\mathrm{Q}}_{/2} - \underset{212}{\mathrm{Q}}_{/1}) = 0 \tag{3.4}$$

not containing A; the condition (3.4) is a linear, homogeneous, *second order* differential identity in ξ_{ik}: it is the well-known St. Venant condition for the plane, as one can immediately verify.

For the Euclidean space, one has to proceed in an analogous way, though more complicated: still the components $\underset{\alpha\beta}{\sigma}$ do not exist ($r = \mathrm{N}$) and (2.20) give nine non-homogeneous conditions, with three arbitrary functions; by choosing cartesian coordinates and $\underset{l}{v}_i = \delta_{li}$, one can reduce them to *six* linear, homogeneous and *second order* differential conditions: they correspond to the components Ω_{ij} of the St. Venant congruence tensor [8, p. 7].

At last, let us consider the simplest case of a non-Euclidean four-manifold, that is a static and spherically symmetric manifold of the Schwarzschild type; there are three linearly independent Killing congruences and the solution $v^i_{(s)}$ of the Killing equation (2.4) is

$$v^i_{(s)} \equiv (a\,;\,0\,;\,b\cos x^3 + c \operatorname{sen} x^3\,;\,d + \operatorname{cotg} x^2\cdot(-\,b \operatorname{sen} x^3 + c\cos x^3)) \tag{3.5}$$

(where a, b, c, d are four arbitrary parameters), if the "spherical" coordinate system (x^0, x^1, x^2, x^3) is used where the metric tensor is [9, Ch. 11]

$$a_{ik} \equiv \operatorname{diag}\cdot(\mathrm{A}(x^1)\,;\,-\,1/\mathrm{A}(x^1)\,;\,-\,(x^1)^2\,;\,-\,(x^1)^2 \operatorname{sen}^2(x^2))\,. \tag{3.6}$$

The congruence conditions can be obtained by the same procedure given in [1], just applied in this section to the Euclidean case; otherwise we recall that the displacement $s_i(\xi_{mn})$ can be obtained more easily with reference to its contravariant components s^i that is if (1.1) is given the form

$$\eta_{ik} \equiv \xi_{ik} - (a_{im}\,s^m_{,k} + a_{km}\,s^m_{,i} + a_{ik,m}\,s^m) = 0. \tag{3.7}$$

In correspondence to any invariant Φ not identically zero (1) and in the chosen coordinate system the component s^1 is

$$s^1 = \delta\Phi/\Phi' \qquad (\Phi' \equiv \mathrm{d}\Phi/\mathrm{d}x^1) \tag{3.8}$$

independent of the choice of Φ, if ξ_{ik} is congruent.

(1) If $\mathrm{A}(x^1) \equiv 1 - \alpha/x^1$ ($\alpha = $ const.), the metric tensor is a solution, if $x^1 \neq 0$, of the field equations $\mathrm{R}_{ik} = 0$; therefore any scalar invariant constructed only by the metric and R_{ik} is identically zero, whereas an invariant constructed by the metric and R_{ikmn} is generally non zero.

Then by the three equations

$$\eta_{10} = 0 \quad ; \quad \eta_{20} = 0 \quad ; \quad \eta_{30} = 0 \tag{3.9}$$

one directly obtains the components s^0 , s^2 , s^3 (with three arbitrary functions), while the *seven* conditions

$$\eta_{\alpha\beta} = 0 \quad ; \quad \eta_{00} = 0 \qquad (\alpha , \beta = 1 , 2 , 3) \tag{3.10}$$

are the searched congruence conditions.

In conclusion, we point out that this procedure allows to calculate the displacement field s_i (whose explicit form is not written here), i.e. the general solution of (1.1), as an affine function of the strain ξ_{ik} and its derivatives, whenever ξ_{ik} is congruent. As for the particular case of the Schwarzschild manifold, a paper with a detailed analysis of the essential congruence conditions corresponding to (3.10) and the displacement field s_i is forthcoming.

4. CONCLUDING REMARKS

In the case of manifolds with rigid motions, the congruence and equilibrium problems discussed in [1] and in the previous sections seem to have some peculiar properties, in contrast with the case of no rigid motions [2].

Firstly, two sets of congruence conditions are "naturally" obtained: by the first one, given by (2.20), the possibility of obtaining a vector field $s_i(\xi_{mn})$ from the strain ξ_{mn} is assured; this field is the very solution of Eq. (1.1) on account of the second set of conditions, obtained by making zero the intrinsic components $\underset{\alpha\beta}{\sigma}$.

As already stated, arbitrary functions may be contained in the first set of conditions, and can be eliminated, in general, only by an explicit use of a particular coordinate mapping; for that, and in contrast with the case of no rigid motions [2], the essential congruence conditions and the most general second-rank symmetric tensor with zero or given divergence cannot be obtained directly in an invariant tensor form.

Even if this method seems less elegant and general than that used by [2], and the actual elimination of the arbitrary functions is somewhat complicated, notwithstanding this procedure and the results obtained may be of some interest, on account of the symmetries of the manifolds analyzed and their physical meaning.

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