Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.2, p. 201–203.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_2_201_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Teoria dei controlli. — *On the closure of the attainable set in infinite dimensional control theory.* Nota di MARIELLA CECCHI BARONI e ANGELA SELVAGGI PRIMICERIO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si considera un problema di controllo in spazi di Banach X, Y relativamente alle condizioni di chiusura per l'insieme raggiungibile in X. Si esamina in particolare il caso: X spazio di Hilbert e $Y = L^{\infty}(O, T : U)$ con U spazio di Hilbert.

1. Let X, Y be two Banach spaces. Following [1], a control process in such spaces is a function x of $(t, u, v) \in [O, T] \times Y \times X$ in X defined by

$$x(t, u, v) = G(t, O)v + \int_0^t G(t, s) B(s) u(s) \, ds \tag{P}$$

where v is a given point of X, $u = (t \to u(t))$ belongs to Y, $(t \to B(t))$ belongs to the space $\mathcal{L}[Y, X]$ of all linear continuous operators from Y to X, and $G : ((t, s) \to G(t, s))$ is the evolution operator generated by some function $A : (t \to A(t))$ of $t \in [O, T]$ into the space of linear (possibly unbounded) operators in X. So G is a function defined for $O \leq s \leq t \leq T$ into $\mathcal{L}[X, X]$ with the properties:

G is strongly continuous in (t, s)

$G(t, s) G(s, r) = G(t, r) \qquad O \leq r \leq s \leq t \leq T$

$G(s, s) =$ the identity of $\mathcal{L}[X, X]$

$\frac{\partial}{\partial t} G(t, s) = A(t) G(t, s) \qquad \left(\frac{\partial}{\partial t} \text{ is the strong derivative}\right)$.

The equality holds a.e. in $[s, T]$ for every $s \in [O, T]$.

Under suitable assumptions ([2]), (P) is the integral form of a linear evolution equation:

$$\dot{x}(t) - A(t) x(t) = B(t) u(t) \qquad \left(\cdot = \frac{d}{dt}\right)$$

with the initial condition: $x(O) = v$.

Let us define the mapping $\Lambda : Y \to X$ by

$$\Lambda u = \int_0^T G(T, s) B(s) u(s) \, ds$$

and consider a subset $\mathfrak{U}$ of Y and the set of attainability:

$$\Gamma = \Lambda \mathfrak{U} + G(T, O) v .$$

(*) Nella seduta del 9 febbraio 1974.

$G(T, O)\,v$ is a point, therefore properties of Γ, in particular its closure can be investigated through the analysis of $\Lambda\mathfrak{A}$.

2. Let U be a Banach space and let $L^p(O, T; U)$ denote the Banach space of all U-valued Bochner strongly measurable functions [4] defined a.e. in $[O, T]$ such that:

$$\|f\| = \left(\int_0^T |f(t)|_U^p \, dt\right)^{1/p} < \infty, \qquad \text{if} \quad 1 < p < \infty$$

$$\|f\| = \text{ess. sup.} \{ |f(t)|_U : \; t \in [O, T] \} < \infty, \qquad \text{if} \quad p = \infty .$$

From now on, we assume:

$$\text{(R)} \qquad L^p(O, T; U) = (L^{p'}(O, T; U'))' \qquad 1 < p \leq \infty .$$

The mapping $\Lambda : u \rightarrow \int_0^T G(T, s)\, B(s)\, u(s)\, ds$ is linear and continuous; consequently $\Lambda\mathfrak{A}$ is a closed bounded convex set of X. if $Y = L^p(O, T; U)$, $1 < p < \infty$, and if $\mathfrak{A}$ is a bounded closed convex set of Y ([1], Th. 2, p. 47).

On the other hand, if $Y = L^\infty(O, T; U)$ the closure of $\Lambda\mathfrak{A}$ in X can be proved, provided that $\mathfrak{A}$ is bounded (convex) and closed in the $L^1(O, T; U')$-topology of $L^\infty(O, T; U)$, and that Λ is continuous in the $L^1(O, T; U')$-topology of $L^\infty(O, T; U)$ and in the weak-topology of X ([1], Th. 2'', p. 48). Such continuity of Λ follows at once if X and U are finite-dimensional Banach spaces. In this note we want to show that Λ is continuous in the same sense when X and U are Hilbert spaces.

3. First, we need some preliminary remarks.

Let X, Y be the conjugate spaces of two Banach spaces: $Y = V^*$, $X = Z^*$. It is easily seen that $\Lambda\mathfrak{A}$ is closed if $\mathfrak{A}$ is bounded (convex) and closed in the V-topology of Y and if Λ is continuous in the V-topology of Y and in the Z-topology of X. This means that $\Lambda = S^*$, where S is a linear strongly continuous mapping from Z to V ([3], 5 sec. 21.5).

Now we can prove the following:

THEOREM. *Let* X, U *be two Hilbert spaces, and* $Y = L^\infty(O, T; U)$. *Let* $B \in L^1(O, T; \mathfrak{L}[U, X])$. *Then, if* $\mathfrak{A}$ *is a subset (convex and) bounded of* $L^\infty(O, T; U)$ *and if* $\mathfrak{A}$ *is closed in the* $L^1(O, T; U)$*-topology of* $L^\infty(O, T; U)$, *the attainable set for the process* (P) *is (bounded convex and) closed in* X.

Proof. Note that, since U is a Hilbert space, (R) holds, i.e.: $L^\infty(O, T; U) = (L^1(O, T; U))'$; so, according to the foregoing remarks, it suffices to prove that $\Lambda = S^*$, where S is a linear continuous mapping from X to $L^1(O, T; U)$. The mapping $(s \rightarrow G(T, s))$ belongs to the space $C(O, T; \mathfrak{L}[X, X])$ of all $\mathfrak{L}[X, X]$-valued continuous functions defined on $[O, T]$. Hence

$(s \to G(T, s) B(s))$ belongs to $L^1(O, T; \mathfrak{L}[U, X])$, and $(s \to B^*(s) G^*(T, s))$ belongs to $L^1(O, T; \mathfrak{L}[X, U])$.

Let us consider the mapping:

$$S: \quad x \to Sx = (s \to B^*(s) G^*(T, s) x)$$

from $x \in X$. Such mapping is linear and continuous and its range is in $L^1(O, T; U)$. In fact:

$$\int_0^T \| B^*(s) G^*(T, s) x \|_U \, ds \leq \| x \|_X \int_0^T \| B^*(s) G^*(T, s) \|_{\mathfrak{L}[X,U]} \, ds .$$

Let us prove that $\Lambda = S^*$, i.e.:

$$\langle Sx, u \rangle = \langle x, \Lambda u \rangle \quad \forall x \in X \quad , \quad \forall u \in (L^1(O, T; U))' = L^\infty(O, T; U) .$$

Since X , U are Hilbert spaces, from the properties of Bochner-integral ([4]), we have:

$$\langle Sx, u \rangle = \int_0^T \langle B^*(s) G^*(T, s) x, u(s) \rangle \, ds =$$

$$= \int_0^T \langle x, G(T, s) B(s) u(s) \rangle \, ds = \langle x, \int_0^T G(T, s) B(s) u(s) \, ds \rangle =$$

$$= \langle x, \Lambda u \rangle \quad \forall x \in X, \quad \forall u \in (L^1(O, T; U))'.$$

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