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On some results on locally power α-set contractions. III

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Topologia. — *On some results on locally power α-set contractions. III.* Nota di VASILE I. ISTRĂŢESCU e ANA I. ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si considerano classi di operatori su spazi di Banach contenenti classi di operatori che ammettono una regolarizzazione generalizzata. Facendo uso della misura di non compattezza di Kuratowski sono dimostrate alcune proprietà di queste classi.

0. In the theory of singular integrals the class of regularizable operators has some applications. Since the regularizability is defined by means of compact operators, the recent progress in the extension of properties of compact operators to larger class of operators, an extension of the notion of regularizability is thus suggested. The purpose of the present paper is to give such an extension.

1. Let X be a Banach space and A be any bounded set in X. By $\alpha(A)$ we denote the infimum of all $\varepsilon > 0$ such that A can be covered by a finite family of subsets with diameter less than ε. The number $\alpha(A)$ is called frequently the Kuratowski number of A or the Kuratowski measure of noncompactness.

A mapping $T : X \to X$ is said to be α–Lipschitz with constant $k \in (0, \infty)$ if for any bounded and noncompact set $A \subset X$

$$\alpha(TA) < k\alpha(A)$$

and if $k \in [0, 1)$ the int is called α–contraction with constant k.

If L is a bounded linear operator on X then it has a regularization if there exists a bounded linear operator S such that

$$SL = I + C$$

where C is a compact operator and S is called the regularizator of the operator T.

DEFINITION 1.1. *A bounded linear operator* L *has a generalized regularization if there exists a linear bounded operator in* S *such that*

$$SL = I + C$$

where C *is an α–contraction.*

(*) Nella seduta del 12 gennaio 1974.

DEFINITION 1.2. *The operator* L *has an equivalent generalized regularization if* L *has a generalized regularization such that*

$$Lu = f \Longleftrightarrow SLf = Sf .$$

for each $f \in X$.

We give now some properties of the above class of operators.

THEOREM 1.3. *If* L *has a generalized regularization then*

$$N(L) = x \quad , \quad Lx = 0$$

is finite dimensional.

Proof. The theorem is proved if we can show that the above set is compact. Indeed, if $\alpha(N(L)) > 0$ we have that

$$\{x , x + Cx = 0 , x \in L , \| x \| = 1\}$$

has the property that

$$\alpha(N(L)) = \alpha C(N(L)) < k\alpha(N(L))$$

which is contradiction and the theorem is proved.

THEOREM 1.4. *If* L *has a generalized regularization then* L *is normally solvable.*

Proof. From the well-known theorem of Hausdorff it is sufficient to prove that the range of L is closed.

Let R(L) be the range of L and let $x_0 \in X$ such that $Lx_0 = f \in R(L)$ and if $x_1 , x_2 , \cdots , x_n$ is a basis for $N(L) = \{x , Lx = 0\}$ then the general solution of the equation

$$Lx = f$$

is of the form

$$x = x_0 + \sum_{i=1}^{n} \alpha_i x_i$$

where α_i are constants. Let

$$\delta = \inf_x \| x \|$$

where x is a solution of the equation $Lx = f$. Then

$$\delta = \lim_{m \to \infty} \left\| x_0 + \sum_{i=1}^{n} \alpha_i^m x_i \right\|$$

and we can choose the sequences $\{\alpha_i^{me}\}$ to be convergent.

Let $\alpha_i^{me} \to \alpha_i^0$ and thus element $\tilde{x}_f = x_0 + \sum_{i=1}^{n} \alpha_i^0 x_i$ has the property that its norm is δ. We wish to show that there exists a constant M such that

(*) $$\| x_f \| \leq M \| f \| .$$

Suppose now that this is not true and thus we find a sequence of elements $\{f_n\}$ such that

$$\left\{\frac{\|\tilde{x}f_n\|}{\|f_n\|}\right\}$$

is not bounded.

Since L has a generalized regularization

$$SL\tilde{x}f_n = \tilde{x}f_n + C\tilde{x}f_n = f_n$$

we have

$$LS\frac{\tilde{x}f_n}{\|xf_n\|} = \frac{\tilde{x}f_n}{\|\tilde{x}f_n\|} + C\frac{\tilde{x}f_n}{\|\tilde{x}f_n\|} = \frac{f_n}{\|\tilde{x}f_n\|} .$$

Since $\left\{\frac{f_n}{\|\tilde{x}f_n\|}\right\}$ converges to zero, we obtain that $\left\{\frac{\tilde{x}f_n}{\|\tilde{x}f_n\|}\right\}$ is relatively compact and we can suppose without loss of generality that $\left\{\frac{\tilde{x}f_n}{\|\tilde{x}f_n\|}\right\}$ is convergent to an element $\tilde{x}_0$. It is clear that, since

$$L\tilde{x}f_n = f_n \quad , \quad L\frac{\tilde{x}f_n}{\|\tilde{x}f_n\|} = \frac{f_n}{\|\tilde{x}f_n\|}$$

the element $\tilde{x}_0$ is in N (L). Thus for each n,

$$L(\tilde{x}f_n - \tilde{x}_0) = f_n$$

and we obtain that

$$\left\|\frac{\tilde{x}f_n}{\|\tilde{x}f_n\|} - x_0\right\| \geq 1 .$$

which is a contradiction and the inequality (*) is proved.

The proof is a contradiction is now as in the case of regularizable operators and for the completeness we give it here.

Let $f_n \in R(L)$ and $\tilde{x}f_n$ be the elements of minimal norm such that $L\tilde{x}f_n = f_n$. Let also $f_n \to f$. By the above inequality $\{\tilde{x}f_n\}$ is bounded and from

$$SL\tilde{x}f_n = \tilde{x}f_n + C\tilde{x}f_n - Sf_n$$

we obtain easily that $\{\tilde{x}f_n\}$ is convergent. This clearly implies that $f = \lim f_n = L(\lim \tilde{x}f_n)$ and the theorem is proved.

The next theorem is an extension of a theorem of Noether. First we recall the notion of the index; the index of the operator T is called the number

$$\text{ind } T = \dim N(T) - \dim N(T)$$

whenever $\dim N(T) < \infty$ or $\dim N(T) < \infty$.

THEOREM 1.5. *If* T *is bounded and has a generalized regularization and ind* $T < \infty$ *then for each compact operator* C

$$\text{ind}(T + C) = \text{ind } T$$

Proof. Since the proof is exactly as in the case of operators with regularization we omit it.

2. Using the notion of generalized regularization some results in the theory of singular integral equations can be extended. We give here a simple result. We use the notion and notation of the book of Mikhlin [3]. (See Ch. VIII, § 35). Let the singular integral equation

$$A_0 u = F^{-1} \varphi(\theta) Fu = g(x)$$

with $g \in L_2(E_m)$ and $\varphi(\theta)^{-1}$ is bounded. The operator $B_0 = F^{-1}(\varphi(\theta)^{-1} F$ is bounded on $L^2(E_m)$. Thus the equation has the solution

$$u = F^{-1}(\varphi(\theta)^{-1}) Fg .$$

We consider the more general equation of the following type

$$F^{-1} \varphi(\theta) Fu + C_1 = g$$

where C_1 is such that

$$B_0 C_1$$

is α–contraction.

An effective example can be constructed as follows: we choose a compact operator $\tilde{C}_1$ and next an α–contraction C_2 such that

$$\| B_0 \| k < 1$$

where k is the constant of the α–contraction C_2. Then

$$C_1 = \tilde{C}_1 + C_2$$

satisfies our condition.

Another application of the above results will be given in our Note [5].

References

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