Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Christian Fabry

A fixed point theorem based on the use of an auxiliary functional

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.2, p. 175–179.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_2_175_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Equazioni funzionali. — *A fixed point theorem based on the use of an auxiliary functional* (*). Nota di CHRISTIAN FABRY, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si presentano teoremi di punti fissi per funzioni non necessariamente continue definite su spazi topologici compatti o su sottoinsiemi debolmente compatti di spazi vettoriali topologici. I risultati generalizzano teoremi ottenuti da W. A. Kirk per applicazioni nonespansive.

1. The aim of this Note is to present a fixed point theorem which provides a generalization for the following results of W. A. Kirk [1, 2].

THEOREM 1. *Suppose* X *is a nonempty, weakly compact, convex subset of a Banach space* E, *and suppose that* X *has normal structure. Then, for every nonexpansive mapping* $T : X \to X$, *there is a point* x *in* X *such that* $x = Tx$.

THEOREM 2. *Suppose* X *is a nonempty, weakly compact subset of a Banach space* E. *If* $T : X \to X$ *is a nonexpansive mapping which has diminishing orbital diameters, then there is a point* x *in* X *such that* $x = Tx$.

Let us recall that a mapping $T : X \to X$ is nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all x, y in X and that it has diminishing orbital diameters if, for each $x \in X$, $x \neq Tx$,

$$\delta(Q(x)) < \infty \qquad \text{and} \qquad \lim_{n \to \infty} \delta(Q(T^n x)) < \delta(Q(x))$$

where $\delta(Q(x))$ is the diameter of the set $Q(x) = \{T^n x ; n = 0, 1, 2, \cdots\}$. On the other hand, a bounded convex subset X in a Banach space E is said to have normal structure if, for every convex subset H of X which contains more than one point, there is a point $x \in H$ which is nondiametral, that is, for which

$$\sup \{\| x - y \| : y \in H\} < \delta(H) = \sup \{\| x - y \| : x, y \in H\}.$$

2. Our first result provides a fixed point existence criterion in the general framework of a compact topological space X; it involves the use of a function $\psi : X \times X \to \mathbf{R}$ (with adequate continuity and boundedness properties) such that $\psi(Tx, Ty) \leq \psi(x, y)$. Other fixed point theorems based on the use of an auxiliary functional have been given by M. Furi and A. Vi-

(*) Lavoro eseguito durante un soggiorno dell'Autore presso l'Istituto di Matematica Applicata. Facoltà d'Ingegneria. Università di Firenze.

(**) Nella seduta del 9 febbraio 1974.

gnoli [3] and by J. H. George, V.M. Sehgal and R. F. Smithson [4] (see also J. W. Thomas [5]); the main difference with the present approach is that the auxiliary functional was defined on X rather than on the product $X \times X$.

THEOREM 3. *Let* X *be a nonempty compact topological space and let* $T : X \to X$ (not necessarily continuous). *Let us assume that a function* $\psi : X \times X \to \mathbf{R}$ *exists such that*

(i) $(\forall x \in X)$, $\psi(x, \cdot)$ *is lower semi-continuous* (l.s.c., in short),

(ii) $(\forall y \in X)$, $\psi(\cdot, y)$ *is* l.s.c.,

(iii) $(\forall x \in X)(\forall y \in X)$, *the set* $\{\psi(x, T^n y);\ n = 0, 1, 2, \cdots\}$ *is bounded in* $\mathbf{R}$,

(iv) $(\forall y \in X)$, *the function* $Q(\cdot, y) = \limsup_{n \to \infty} \psi(\cdot, T^n y)$ *is* l.s.c.,

(v) $(\forall x \in X)(\forall y \in X)$, $\psi(Tx, Ty) \leq \psi(x, y)$,

(vi) $(\forall H \subset X : H$ *closed*, $T(H) \subset H$, H *containing more than one point*) $(\exists x \in H, y \in H)$, $\limsup_{n \to \infty} \psi(x, T^n y) < \sup\{\psi(x, y) : x \in H, y \in H\}$.

Then there is a point $x \in X$ *such that* $x = Tx$.

Proof. 1) Let Φ be the family of closed and nonempty subsets H of X such that $T(H) \subset H$ (invariant sets). The family Φ is nonempty, since $X \in \Phi$, and can obviously be (partially) ordered by inclusion. It then results from the compactness of X and from Zorn's lemma that Φ contains a minimal element, say K.

2) For $y \in X$, let $K_0(y) = \{x \in K : Q(x, y) = \inf_{u \in K} Q(u, y)\}$; as $Q(\cdot, y)$ is l.s.c., the set $K_0(y)$ is nonempty and closed. Moreover, it is invariant since $Q(Tx, y) \leq Q(x, y)$ for $x \in X$, $y \in X$. By the minimality of K, it then results that, $\forall y \in X$, $K_0(y) = K$ and that $Q(x, y)$ is independent of $x \in K$. Let us then introduce the function $r : X \to \mathbf{R}$ defined by $r(y) = Q(x, y)$, x being any element of K.

3) Let $r_0 = \inf\{r(y) : y \in K\}$; it is clear that

$$(\forall \varepsilon > 0)(\exists y \in K)(\forall x \in K)(\exists N \geq 0)(\forall n \geq N),\ \psi(x, T^n y) \leq r_0 + \varepsilon .$$

The set $S(x, \varepsilon) = \{y \in K : \psi(x, y) \leq r_0 + \varepsilon\}$ is nonempty and closed; as K is compact and as the family $\{S(x, \varepsilon) : x \in K, \varepsilon > 0\}$ has the finite intersection property, the intersection $S = \bigcap_{\varepsilon > 0} \bigcap_{x \in K} S(x, \varepsilon)$ is nonempty and closed.

4) It will be shown now that $T(S) \subset S$ which will imply, by the minimality of K, that $S = K$. Indeed, $\forall x \in K$, $\forall y \in S$, the following relations hold:

$$\psi(Tx, Ty) \leq \psi(x, y) \leq r_0$$

and, therefore, $\sup\{\psi(x, Ty) : x \in T(K)\} \leq r_0$, for all $y \in S$. But, the closure cl T(K) of T(K) is invariant because $T(K) \subset \mathrm{cl}\, T(K) \subset K$, which implies that $\mathrm{cl}\, T(K) = K$. As $\psi(\cdot, y)$ is l.s.c., it then follows that $\sup\{\psi(x, Ty) : x \in K\} \leq r_0$ or that $Ty \in S$, for all $y \in S$.

5) For all $x \in K, y \in K$, the following relation then holds:

$$\psi(x, y) \leq r_0 = \inf\{\limsup_{n\to\infty} \psi(x, T^n y) : x, y \in K\};$$

this relation, together with hypothesis (vi), shows that K must consist of a single point.

3. A similar result can be given when X is a weakly compact subset of a locally convex topological vector space (LCTVS) E. In that case, it is possible, as shown by the following lemma, to make appropriate hypotheses about the function $\psi(\cdot, y) : X \to \mathbf{R}$ in order to get the required properties of semi-continuity for the function $Q(\cdot, y) = \limsup_{n\to\infty} \psi(\cdot, T^n y)$. Let us recall that a function $\psi : E \to \mathbf{R}$ is called quasi-convex if the sets $P(r) = \{y \in E : \psi(y) \leq r\}$, $r \in \mathbf{R}$, are convex.

LEMMA. *Let* E *be a locally convex topological vector space. If a family* $\{\psi_n : n = 0, 1, 2, \cdots\}$ *of real quasi-convex functions defined on* E *is bounded at each point of* E *and is equicontinuous, then the function* $Q : E \to \mathbf{R} : x \to \limsup_{n\to\infty} \psi_n(x)$ *is quasi-convex and* l.s.c., *and thus weakly* l.s.c.

Proof. It must be shown that the sets $P(r) = \{x \in E : Q(x) \leq r\}$, $r \in \mathbf{R}$, are convex and closed, and thus weakly closed.

1) Let us prove first that, $\forall r \in \mathbf{R}$, the set $P(r)$ is closed.

Let $x \in \mathrm{cl}\, P(r)$; the equicontinuity of $\{\psi_n : n = 0, 1, 2, \cdots\}$ implies that

$$(\forall \varepsilon > 0)\,(\exists u \in P(r))\,(\forall n \geq 0),\ |\psi_n(x) - \psi_n(u)| \leq \varepsilon/2 .$$

On the other hand, it results from the definition of Q that

$$(\forall \varepsilon > 0)\,(\forall u \in E)\,(\exists N \geq 0)\,(\forall n \geq N),\ \psi_n(u) - Q(u) \leq \varepsilon/2 .$$

Therefore, it is clear that

$$(\forall \varepsilon > 0)\,(\exists N' \geq 0)\,(\forall n \geq N'),\ \psi_n(x) \leq r + \varepsilon ,$$

which shows that $Q(x) \leq r$ and that $P(r)$ is closed.

2) Let $x_1, x_2 \in P(r)$ and let us show that any point x belonging to the segment $[x_1, x_2]$ is also in $P(r)$. By hypothesis, $Q(x_i) \leq r$ for $i = 1, 2$ and, therefore,

$$(\forall \varepsilon > 0)\,(\exists N_i \geq 0)\,(\forall n \geq N_i),\ \psi_n(x_i) \leq r + \varepsilon , \qquad (i = 1, 2);$$

taking $N = \max\{N_1, N_2\}$ and using the quasi-convexity of ψ, one obtains

$$(\forall \varepsilon > 0)\,(\exists N \geq 0)\,(\forall n \geq N)\,,\ \psi_n(x) \leq r + \varepsilon$$

or $Q(x) \leq r$ for $x \in [x_1, x_2]$.

We are now able to state our main result. In the theorem, X is a weakly compact subset of a LCTVS E and Φ denotes the family of nonempty invariant subsets of X which are intersections of X with closed convex subsets of E.

THEOREM 4. *Let* X *be a nonempty weakly compact subset of a* LCTVS E *and let* $T : X \to X$ (not necessarily continuous). *Let us assume that a function* $\psi : E \times E \to \mathbf{R}$ *exists such that*

(i) $(\forall x \in X)\,,\ \psi(x, \cdot)$ *is quasi-convex and* (strongly) l.s.c. *on* E,

(ii) $(\forall y \in X)\,,\ \psi(\cdot, y)$ *is quasi-convex on* E,

(iii) *the family of functions* $\{\psi(\cdot, y) : y \in X\}$ *is* (strongly) *equicontinuous on* E,

(iv) $(\forall x \in E)\,(\forall y \in X)$, *the set* $\{\psi(x, T^n y) : n = 0, 1, 2, \cdots\}$ *is bounded in* $\mathbf{R}$.

(v) $(\forall x \in X)\,(\forall y \in X)\,,\ \psi(Tx, Ty) \leq \psi(x, y)$,

(vi) $(\forall K \in \Phi$, K *containing more than one point*) $(\exists x \in K, y \in K)$, $\limsup_{n \to \infty} \psi(x, T^n y) < \sup\{\psi(x, y) : x, y \in K\}$.

Then, there is a point $x \in X$ *such that* $x = Tx$.

Proof. The proof follows very closely that of Theorem 3 and will be given only in its broad lines; the notation will be similar to that of Theorem 3.

1) It is clear that Φ has a minimal element K, a set $H \in \Phi$ being closed for the topology induced on X by the weak topology of E.

2) For $y \in X$, let again $K_0(y) = \{x \in K : Q(x, y) = \inf_{u \in K} Q(u, y)\}$; as $Q(\cdot, y)$ is l.s.c. and quasi-convex (lemma), the set $K_0(y)$ is nonempty and is the intersection of X with a closed convex subset of E. Moreover, $K_0(y)$ is invariant, and therefore $K_0(y) = K$. As in theorem 3, a function $r : X \to \mathbf{R}$ can be introduced by the definition $r(y) = Q(x, y)$, x being any element of K.

3) If $r_0 = \inf\{r(y) : y \in K\}$, the sets $S(x, \varepsilon) = \{y \in K : \psi(x, y) \leq r_0 + \varepsilon\}$ where $x \in K$, $\varepsilon > 0$ are nonempty and are intersections of X with closed convex subsets of E. The same will be true for $S = \bigcap_{\varepsilon > 0} \bigcap_{x \in K} S(x, \varepsilon)$.

4) It can be shown that S is invariant and thus that $S = K$. Indeed, it is clear that $\sup\{\psi(x, Ty) : x \in T(K)\} \leq r_0$. But, the intersection of K with the convex closure $\operatorname{cocl} T(K)$ of $T(K)$ is invariant; it then results that $K = K \cap \operatorname{cocl} T(K)$ and that $\sup\{\psi(x, Ty) : x \in K\} \leq r_0$.

5) The conclusion then follows as in Theorem 3.

In fact, the function ψ need not be defined on the whole product $E \times E$, but only on cocl $X \times$ cocl X, the hypotheses being then modified accordingly. On the other hand, it can be seen that, when E is a Banach space and ψ is defined by $\psi(x, y) = \| x - y \|$, Theorems 1 and 2 are special cases of Theorem 4, as hypothesis (vi) may be replaced by either one of the following two conditions:

(vi′) $(\forall K \in \Phi, K$ containing more than one point$)$ $(\exists x \in K)$,
$$\sup \{ \psi(x, y) : y \in K \} < \sup \{ \psi(x, y) : x, y \in K \},$$

(vi″) $(\forall x \in X, x \neq Tx)$, $$\lim_{N \to \infty} [\sup \{ \psi(T^m x, T^n x) : m, n \geq N \}] < \sup \{ \psi(T^m x, T^n x) : m, n \geq 0 \}.$$

Indeed, the first condition obviously implies (vi) whereas the second one implies that, for any invariant subset K of X, either K contains a fixed point or there exists $x \in K$, $N \geq 0$ such that

$$\sup \{ \psi(T^N x, T^n x) : n \geq N \} = \sup \{ \psi(T^m x, T^n x) : m, n \geq N \} < \sup \{ \psi(x, y) : x, y \in K \}.$$

REFERENCES

[1] W. A. KIRK, *A fixed point theorem for mappings which do not increase distances*, «Amer. Math. Monthly», *72*, 1004–1006 (1965).

[2] W. A. KIRK, *Fixed point theorems for nonexpansive mappings*, «Proc. A. M. S. Symp. Pure Math.», *18*, 162–168 (1970).

[3] M. FURI and A. VIGNOLI, *A remark about some fixed point theorems*, «Boll. U. M. I.», *2*, 197–200 (1970).

[4] J. H. GEORGE, V. M. SEHGAL and R. F. SMITHSON, *Application of Liapunov's direct method to fixed point theorems*, «Proc. A.M.S.», *28*, 613–620 (1971).

[5] J. W. THOMAS, *On the Liapunov type fixed point theorem*, «Fund. Math.», *78*, 141–143 (1973).