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Boundedness of solutions of perturbed systems of differential equations

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Equazioni differenziali ordinarie. — *Boundedness of solutions of perturbed systems of differential equations.* Nota di RODNEY S. RAMBALLY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore considera il sistema $x'=f(t,x)$ e il sistema perturbato $y'=f(t,y)+g(t,y)$ e dà condizioni sufficienti atte ad assicurare la limitatezza delle soluzioni del sistema perturbati. In particolare è considerato il caso in cui $f(t,y)$ è lineare.

1. We consider the systems of differential equations

$$x' = f(t, x) \tag{1}$$

$$y' = f(t, y) + g(t, y) \tag{2}$$

where x, y, f and g are n-dimensional vectors and where $f(t, 0) = 0$, thereby making $x = 0$ a solution of (1). Let $|\cdot|$ denote a convenient vector norm and a corresponding matrix norm.

Assume there exists a solution $x(t)$ of (1) which is defined for all $t \geq t_0$. Recall that the solution $x = 0$ of (1) is said to be

a) *uniformly stable* if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x(t)$ satisfies $|x(t_1)| < \delta$ for some $t_1 \geq t_0$, then $|x(t)| < \varepsilon$ for all $t \geq t_1$;

b) *uniformly asymptotically stable* if, in addition to (a), there is a $\delta > 0$ and for each $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that whenever $|x(t_1)| < \delta$ for some $t_1 \geq t_0$, then $|x(t)| < \varepsilon$ for all $t \geq t_1 + T$.

We shall usually denote the solution of (1) through (t_0, x_0) by $x(t, t_0, x_0)$ and similarly the solution of (2) through (t_0, y_0) by $y(t, t_0, y_0)$.

2. For the first theorem we assume that f satisfies a uniform Lipschitz condition on $D_M = \{(t, x) : t \geq 0, |x| \leq M, M > 0\}$. Let L be the Lipschitz constant.

THEOREM 1. *Let* $0 \leq t < \infty$ *and* $0 < r < H$. *Assume there exists* $\alpha > 0$ *such that whenever* $|y| \leq \alpha$, $|g(t, y)| \leq \omega(t, |y|)$, *where* $\omega(t, r)$ *is a continuous, scalar function which is monotone, non-decreasing in r for each fixed t and which satisfies*

$$\int_a^\infty \omega(t, c)\, dt < \infty \qquad \textit{for} \quad 0 < c < H, \qquad 0 \leq a < \infty.$$

Assume that the solution $x = 0$ *of* (1) *is uniformly asymptotically stable. Then for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and* $T > 0$ *such that if* $t_0 \geq T$ *and* $|y_0| < \delta$, *then* $|y(t, t_0, y_0)| < \varepsilon$ *for all* $t \geq t_0$.

(*) Nella seduta del 9 febbraio 1974.

We need the following lemma in order to prove the theorem.

LEMMA I. *Let* $x(t, \bar{t}_0, \bar{y}_0)$ *and* $y(t, \bar{t}_0, \bar{y}_0)$ *be solutions of* (1) *and* (2) *respectively. Assume that* $|y(t, \bar{t}_0, \bar{y}_0)| \leq \alpha$ *for* $t \in [\bar{t}_0, \bar{t}_0 + \tau]$ *for some* $\tau > 0$. *Then*

$$|y(t, \bar{t}_0, \bar{y}_0) - x(t, \bar{t}_0, \bar{y}_0)| \leq e^{L\tau} \int_{\bar{t}_0}^{t} \omega(s, \alpha)\, ds$$

for $t \in [\bar{t}_0, \bar{t}_0 + \tau]$.

Proof. For simplicity, we denote $y(s, \bar{t}_0, \bar{y}_0)$ and $x(s, \bar{t}_0, \bar{y}_0)$ by $y(s)$ and $x(s)$ respectively. Now

$$|y(t, \bar{t}_0, \bar{y}_0) - x(t, \bar{t}_0, \bar{y}_0)|$$

$$= \Bigg| \bar{y}_0 + \int_{\bar{t}_0}^{t} [f(s, y(s)) + g(s, y(s))]\, ds$$

$$- \bar{y}_0 - \int_{\bar{t}_0}^{t} f(s, x(s))\, ds \Bigg|$$

$$= \left| \int_{\bar{t}_0}^{t} [f(s, y(s)) - f(s, x(s)) + g(s, y(s))]\, ds \right|$$

$$\leq \int_{\bar{t}_0}^{t} L\, |y(s) - x(s)|\, ds + \int_{\bar{t}_0}^{t} \omega(s, |y(s)|)\, ds .$$

Hence (Brauer [1], Lemma 2)

$$|y(t, \bar{t}_0, \bar{y}_0) - x(t, \bar{t}_0, \bar{y}_0)|$$

$$\leq \int_{\bar{t}_0}^{t} \omega(s, |y(s)|)\, e^{L(t-s)}\, ds$$

$$\leq e^{L\tau} \int_{\bar{t}_0}^{t} \omega(s, \alpha)\, ds .$$

Proof of Theorem I. Let all constants corresponding to (1) be starred. Let $0 < \varepsilon \leq \alpha$. Choose $\delta = \delta(\varepsilon) = \delta^*(\varepsilon/2)$ (where δ^* is as in the definition of uniform stability of the zero solution of (1)) such that $\delta < \varepsilon$. Let $\tau = \tau(\varepsilon) = T^*(\delta/2)$ and choose T large enough such that

$$\int_{T}^{\infty} \omega(s, \alpha)\, ds < \frac{\delta}{2\, e^{L\tau}} .$$

Finally choose $t_0 \geq T$ and $|y_0| < \delta$. Note that for such a y_0, $|x(t, t_0, y_0)| < \varepsilon/2$ for $t \geq t_0$ and $|x(t, t_0, y_0)| < \delta/2$ for $t \geq t_0 + T^*(\delta/2)$. Provided that $|y(t, t_0, y_0)| < \varepsilon \leq \alpha$ for $t \in [t_0, t_0 + \tau]$, we have, using Lemma 1,

$$|y(t, t_0, y_0)| \leq |y(t, t_0, y_0) - x(t, t_0, y_0)| + |x(t, t_0, y_0)|$$

$$\leq e^{L\tau} \int_{t_0}^{t} \omega(s, \alpha)\, ds + \frac{\varepsilon}{2}$$

$$\leq e^{L\tau} \int_{T}^{t} \omega(s, \alpha)\, ds + \frac{\varepsilon}{2}$$

$$< \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon .$$

Thus $y(t, t_0, y_0)$ can be continued to the whole of $[t_0, t_0 + \tau]$ and on this interval $|y(t, t_0, y_0)| < \varepsilon$. Let $y_1 = y(t_0 + \tau, t_0, y_0)$. Then

$$|y_1| \leq |y_1 - x(t_0 + \tau, t_0, y_0)| + |x(t_0 + \tau, t_0, y_0)|$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta .$$

Assume that for some $m > 0$, $|y(t, t_0, y_0)| < \varepsilon$ for $t \in [t_0, t_0 + m\tau]$ and that $|y(t_0 + m\tau, t_0, y_0)| < \delta$. Let $y_m = y(t_0 + m\tau, t_0, y_0)$. We would like to show that $|y(t, t_0, y_0)| < \varepsilon$ for $t \in [t_0 + m\tau, t_0 + (m + 1)\tau]$. Let us consider $y(t, t_0 + m\tau, y_m)$ on $[t_0 + m\tau, t_0 + (m + 1)\tau]$. Provided that $|y(t, t_0 + m\tau, y_m)| < \varepsilon$ on $[t_0 + m\tau, t_0 + (m + 1)\tau]$ we have

$$|y(t, t_0 + m\tau, y_m)| \leq |y(t, t_0 + m\tau, y_m) - x(t, t_0 + m\tau, y_m)|$$

$$+ |x(t, t_0 + m\tau, y_m)|$$

$$\leq e^{L\tau} \int_{t_0 + m\tau}^{t} \omega(s, \alpha)\, ds + \frac{\varepsilon}{2}$$

$$\leq e^{L\tau} \int_{T}^{t} \omega(s, \alpha)\, ds + \frac{\varepsilon}{2}$$

$$< \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon .$$

Thus $|y(t, t_0, y_0)| < \varepsilon$ on the entire interval $[t_0 + m\tau, t_0 + (m + 1)\tau]$. Finally, let $y_{m+1} = y(t_0 + (m + 1)\tau, t_0, y_0)$. Then

$$|y_{m+1}| \leq |y(t_0 + (m + 1)\tau, t_0 + m\tau, y_m)$$

$$- x(t_0 + (m + 1)\tau, t_0 + m\tau, y_m)|$$

$$+ |x(t_0 + (m + 1)\tau, t_0 + m\tau, y_m)| .$$

This follows because $y(t, t_0, y_0)$ can be continued to $t_0 + (m+1)\tau$, thereby making $y(t_0 + (m+1)\tau, t_0, y_0)$ and $y(t_0 + (m+1)\tau, t_0 + m\tau, y_m)$ equal. Thus

$$|y_{m+1}| < \frac{\delta}{2} + \frac{\delta}{2} = \delta .$$

Since $|y(t, t_0, y_0)| < \varepsilon$ on every interval $[t_0 + m\tau, t_0 + (m+1)\tau]$, then $|y(t, t_0, y_0)| < \varepsilon$ on $[t_0, \infty)$. The proof of the theorem is now complete.

We shall now consider the case when our homogeneous system (1) is linear. In Theorem 2, we study the systems

$$x' = A(t)\, x \tag{3}$$

$$y' = A(t)\, y + f(t, y) + b(t), \tag{4}$$

where $A(t)$ and f are continuous. Let $X(t)$ be a fundamental matrix of (3) and let X_1 be the subspace of points in R^n which are values for $t = 0$ of the bounded solutions of (3). Let X_2 be a fixed subspace complementary to X_1 and let P_1 and P_2 be the corresponding projections of R^n onto X_1 and X_2.

THEOREM 2. *Assume that*

$$\int_0^t |X(t)\, P_1\, X^{-1}(s)|^q\, ds + \int_t^\infty |X(t)\, P_2\, X^{-1}(s)|^q\, ds \le K$$

for some positive constant K. *Let $f(t, y)$ be a continuous function such that*

$$|f(t, y_1) - f(t, y_2)| \le \gamma\, |y_1 - y_2| \quad \textit{where} \quad \gamma K < 1,$$

$0 \le t < \infty$, $|y_1| \le \beta$, $|y_2| \le \beta$ *for some constant* β. *Assume that* $|b(t)| \in L^p[0, \infty)$, *where* $1 < p < \infty$ *and* $1/p + 1/q = 1$ *and let the* L^p *norm of* $|b(t)|$, $|b|_p$, *be such that*

$$|b|_p \le \frac{\beta(1 - \gamma K)}{2 K^{1/q}} .$$

Then there exists a unique solution $y(t)$ of (4) such that $|y(t)| \le \beta$.

Proof. Let $y(t)$ be any continuous function such that $|y(t)| \le \beta$ and let

$$\tau y(t) = \int_0^t X(t)\, P_1\, X^{-1}(s)\, (b(s) + f(s, y(s)))\, ds$$

$$- \int_t^\infty X(t)\, P_2\, X^{-1}(s)\, (b(s) + f(s, y(s)))\, ds.$$

We claim that τ is a contraction mapping. This follows because

$$\begin{aligned}
|\tau y_1 - \tau y_2| = \Big| \int_0^t & X(t)\, P_1\, X^{-1}(s)\, (b(s) + f(s, y_1(s)))\, ds \\
& - \int_t^\infty X(t)\, P_2\, X^{-1}(s)\, (b(s) + f(s, y_1(s)))\, ds \\
& - \int_0^t X(t)\, P_1\, X^{-1}(s)\, (b(s) + f(s, y_2(s)))\, ds \\
& + \int_t^\infty X(t)\, P_2\, X^{-1}(s)\, (b(s) + f(s, y_2(s)))\, ds \Big| \\
& = \Big| \int_0^t X(t)\, P_1\, X^{-1}(s)\, (f(s, y_1(s)) - f(s, y_2(s)))\, ds \\
& + \int_t^\infty X(t)\, P_2\, X^{-1}(s)\, (f(s, y_2(s)) - f(s, y_1(s)))\, ds \Big| \\
& \leq \int_0^t |X(t)\, P_1\, X^{-1}(s)|\, |f(s, y_1(s)) - f(s, y_2(s))|\, ds \\
& + \int_t^\infty |X(t)\, P_2\, X^{-1}(s)|\, |f(s, y_1(s)) - f(s, y_2(s))|\, ds \\
& \leq \int_0^t |X(t)\, P_1\, X^{-1}(s)|\, \gamma\, |y_1(s) - y_2(s)|\, ds \\
& + \int_t^\infty |X(t)\, P_2\, X^{-1}(s)|\, \gamma\, |y_1(s) - y_2(s)|\, ds \\
& \leq |y_1 - y_2|\, \gamma K.
\end{aligned}$$

We wish to prove that $|\tau y(t)| \leq \beta$. Now

$$\begin{aligned}
|\tau y(t)| = \Big| \int_0^t & X(t)\, P_1\, X^{-1}(s)\, (b(s) + f(s, y(s)))\, ds \\
& - \int_t^\infty X(t)\, P_2\, X^{-1}(s)\, (b(s) + f(s, y(s)))\, ds \Big|
\end{aligned}$$

$$\leq \int_0^t |X(t)P_1X^{-1}(s)b(s)|\,ds + \int_0^t |X(t)P_1X^{-1}(s)f(s,y(s))|\,ds$$

$$+ \int_t^\infty |X(t)P_2X^{-1}(s)b(s)|\,ds + \int_t^\infty |X(t)P_2X^{-1}(s)f(s,y(s))|\,ds$$

$$\leq \Big(\int_0^t |X(t)P_1X^{-1}(s)|^q\,ds\Big)^{1/q} |b|_p + \gamma\beta \int_0^t |X(t)P_1X^{-1}(s)|\,ds$$

$$+ \Big(\int_t^\infty |X(t)P_2X^{-1}(s)|^q\,ds\Big)^{1/q} |b|_p + \gamma\beta \int_t^\infty |X(t)P_2X^{-1}(s)|\,ds.$$

This last inequality was obtained by applying the Hölder inequality. We now conclude that

$$|\tau y(t)| \leq |b|_p\, 2\,K^{1/q} + \gamma\beta K \leq \beta.$$

By the contraction principle, the integral equation $y = \tau y$ has a unique solution $y(t)$ such that $|y(t)| \leq \beta$. By differentiation, we see that this $y(t)$ is a solution of (4).

Let us now assume that our perturbed linear system is

$$y' = A(t)\,y + f(t,y), \tag{5}$$

where $|f(t,y(t))| \in L^p[0,\infty)$. Let the L^p norm of $f(t,y)$ for $|y| \leq \beta$ be denoted by $|f|_p^\beta$ and let $X(t)$, P_1 and P_2 be as previously described.

THEOREM 3. *Assume that*

$$\Big(\int_0^t |X(t)P_1X^{-1}(s)|^q\,ds\Big)^{1/q} + \Big(\int_t^\infty |X(t)P_2X^{-1}(s)|^q\,ds\Big)^{1/q} \leq K.$$

Let $x(t)$ be a solution of (3) *such that* $|x(t)| \leq \alpha$ *for some constant* α *and assume that*

$$|f|_p^{2\alpha} \leq \frac{\alpha}{K}\cdot$$

Then there exists a solution $y(t)$ of (5) *such that* $|y(t)| \leq 2\alpha$.

Proof. For $|y(t)| \leq 2\alpha$ define the integral operator G by

$$G(y(t)) = x(t) + \int_0^t X(t)P_1X^{-1}(s)f(s,y(s))\,ds$$

$$- \int_t^\infty X(t)P_2X^{-1}(s)f(s,y(s))\,ds.$$

Then

$$
\begin{aligned}
|G(y(t))| &\leq |x(t)| + \int_0^t |X(t)\,P_1\,X^{-1}(s)|\,|f(s\,,y(s))|\,ds \\
&+ \int_t^\infty |X(t)\,P_2\,X^{-1}(s)|\,|f(s\,,y(s))|\,ds \\
&\leq \alpha + \left(\int_0^t |X(t)\,P_1\,X^{-1}(s)|^q\,ds\right)^{1/q} |f|_p^{2\alpha} \\
&+ \left(\int_t^\infty |X(t)\,P_2\,X^{-1}(s)|^q\,ds\right)^{1/q} |f|_p^{2\alpha} \\
&\leq \alpha + |f|_p^{2\alpha}\,K \\
&\leq \alpha + \alpha = 2\,\alpha .
\end{aligned}
$$

Thus Gy is uniformly bounded for all $y(t)$ such that $|y(t)| \leq 2\,\alpha$. The continuity of f can be used to establish the continuity of G. Now Gy is a solution of

$$
v' = A(t)\,v + f(t\,,y(t)). \tag{6}
$$

since $x(t)$ is a solution of (3). Hence Gy has a uniformly bounded derivative on finite intervals, from which it follows that $\{Gy\}$ is equicontinuous. All hypotheses of Schauder's fixed point theorem are satisfied and hence the equation $Gy = y$ has a solution $y(t)$ such that $|y(t)| \leq 2\,\alpha$. Because of (6), this $y(t)$ is also a solution of (5).

Remark. To handle the case when $p = 1$ we assume that

$$
|X(t)\,P_1\,X^{-1}(s)| \leq K\,, \qquad 0 \leq s \leq t\,,
$$

$$
|X(t)\,P_2\,X^{-1}(s)| \leq K\,, \qquad 0 \leq t \leq s.
$$

The proof remains almost the same.

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