Djordje S. Djukic

Whittaker's equations of non-holonomic mechanical systems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.1, p. 55–61.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_1_55_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1974.

Meccanica. — *Whittaker's equations of non-holonomic mechanical systems* (*). Nota di DJORDJE S. DJUKIĆ presentata (**) dal Socio B. FINZI.

RIASSUNTO. — Il suddetto estratto rappresenta la diminuzione delle equazioni differenziali del moto del sistema meccanico anolonomo per mezzo dell'energia integrale. Le equazioni differenziali del moto sono in effetti le equazioni Hamel–Boltzmann date nelle quasi–coordinate. Le Whittaker equazioni esequite sono applicate su un problema semplice della meccanica anolonoma.

I. INTRODUCTION

Whittaker (see [1] or [2], p. 64) has shown that the energy integral can be used for the reduction of a given holonomic conservative dynamical system with n degrees of freedom to another dynamical system with only $(n - 1)$ degrees of freedom. In these equations, the so-called Whittaker equations, a generalized coordinate plays the role of time as the independent variable. The solution of the equations yields the trajectories of the motion.

The aim of this article is to extend Whittaker's idea to non-holonomic mechanical systems which possesses an energy integral.

Let us consider a non-holonomic dynamical system. In this system the number of independent coordinates $(q_1, \cdots, q_n)$ required in order to specify the configuration of the system at any time is greater than the number of degrees of freedom $(n - l)$ of the system, owing to the fact that the system is subject to l constraints, which are expressed by the number of l non-integrable kinematical relations of the form

$$\sum_{k=1}^{n} a_{sk} \dot{q}_k = 0 \qquad (s = 1, \cdots, l), \tag{1}$$

where a_{sk} are given functions of the generalized coordinates $(q_1, \cdots, q_n)$.

For such mechanical systems the governing equations of motion expressed in terms of quasi-coordinates, the so-called Hamel-Boltzmann equations (see for example [5], p. 370), are

$$\frac{d}{dt}\frac{\partial L}{\partial \omega_s} + \sum_{r=1}^{n} \sum_{t=l+1}^{n} \gamma_{ts}^{r} \frac{\partial L}{\partial \omega_r} \omega_t - \frac{\partial L}{\partial \pi_s} = P_s^*, \quad (s = l + 1, \cdots, n). \tag{2}$$

(*) Department of Mechanics, Mechanical Engineering Faculty, University of Novi Sad, 21000 Novi Sad, Yugoslavia.

(**) Nella seduta del 12 gennaio 1974.

Here: $L = T - \Pi$–Lagrange's function; T-kinetic energy; Π–potential energy; P_s^*–non-conservative forces; π_s–quasi-coordinates.

The quasi-velocities $\omega_s (s = 1, \cdots, n)$ are linear combinations of the velocities, and they are selected in such a way that the first l of them are equal to zero as follows from (1).

$$\omega_s = \frac{d\pi_s}{dt} = \sum_{k=1}^{n} a_{sk}\, \dot{q}_k = 0, \qquad (s = 1, \cdots, l), \tag{3}$$

$$\omega_e = \frac{d\pi_e}{dt} = \sum_{k=1}^{n} a_{ek}\, \dot{q}_k \qquad (e = l + 1, \cdots, n), \tag{4}$$

where the $a_{ek} (e = l + 1, \cdots, n\,;\, k = 1, \cdots, n)$ are given functions of $q_1, \cdots, q_n$. Solving (3–4) with respect to the $\dot{q}_k$ we have the equations

$$\dot{q}_s = \sum_{k=l+1}^{n} b_{sk}\, \omega_k \qquad (s = 1, \cdots, n), \tag{5}$$

where the b_{sk} are functions of $q_1, \cdots, q_n$.

These n equations together with the $n - l$ equations (2) forms the system of $2n - l$ equations of the first order whose solution furnishes n quantities q_s and $n - l$ unknown quasi-velocities ω_e.

The so-called objects of anholonomity γ_{tm}^{s} have the structure

$$\gamma_{tm}^{s} = \sum_{k=1}^{n} \sum_{r=1}^{n} \left(\frac{\partial a_{sk}}{\partial q_r} - \frac{\partial a_{sr}}{\partial q_k} \right) b_{rt}\, b_{km}, \tag{6}$$

and

$$\frac{\partial}{\partial \pi_s} = \sum_{r=1}^{n} b_{rs} \frac{\partial}{\partial q_r} \qquad (s, t, m = 1, \cdots, n). \tag{7}$$

If Lagrange's function does not explicitly depend on time, i.e. $\partial L/\partial t = 0$, and if the non-conservative forces are equal to zero ($P_s^* = 0$) this mechanical system possesses an energy integral in the form (see for example [3], p. 200)

$$\sum_{s=l+1}^{n} \omega_s \frac{\partial L}{\partial \omega_s} - L = h = \text{const.} \tag{8}$$

2. Whittaker's equations for a non-holonomic mechanical system

At this point we generalize Whittaker's ideas [1] to non-holonomic conservative mechanical systems.

Let us suppose that the quasi-coordinate π_n plays the role of time as the independent variable. Than we have the following relation

$$\omega_p = \omega_n\, \pi_p' \qquad (p = 1, \cdots, n - 1), \tag{9}$$

where $\pi_p' = d\pi_p/d\pi_n$.

If we replace in the Lagrange function L the quasi-velocities ω_m by (9) we have the function Ω, i.e.

(10) $$L(\pi_1, \cdots, \pi_n, \omega_1, \cdots, \omega_n) = \Omega(\pi_1, \cdots, \pi_n, \omega_n, \pi_1', \cdots, \pi_{n-1}').$$

Then differentiating this expression one obtains

(11) $$\frac{\partial L}{\partial \omega_n} = \frac{\partial \Omega}{\partial \omega_n} - \sum_{p=1}^{n-1} \frac{\omega_p}{\omega_n^2} \frac{\partial \Omega}{\partial \pi_p'},$$

(12) $$\frac{\partial L}{\partial \omega_p} = \frac{1}{\omega_n} \frac{\partial \Omega}{\partial \pi_p'}, \qquad (p = 1, \cdots, n-1)$$

(13) $$\frac{\partial L}{\partial \pi_m} = \frac{\partial \omega}{\partial \pi_m}, \qquad (m = 1, \cdots, n).$$

Combining (11) and (12) we have

(14) $$\frac{\partial \Omega}{\partial \omega_n} = \frac{\partial L}{\partial \omega_n} + \sum_{p=1}^{n-1} \frac{\partial L}{\partial \omega_p} \frac{\omega_p}{\omega_n}.$$

Using this equation the equation of energy (8) may be written in the form

(15) $$\omega_n \frac{\partial}{\partial \omega_n} \Omega(\pi_m, \omega_n, \pi_p') - \Omega(\pi_m, \omega_n, \pi_p') = h,$$
$$(m = 1, \cdots, n;\ p = 1, \cdots, n-1)$$

where (15) furnishes ω_n as a function of the quantities $\pi_1, \cdots, \pi_n, \pi_1', \cdots, \pi_{n-1}'$

(16) $$\omega_n = \omega_n(\pi_1, \cdots, \pi_n, \pi_1', \cdots, \pi_{n-1}', h).$$

Substituting for ω_n in the function $\partial\Omega/\partial\omega_n$, we obtain the function L'

(17) $$L'(\pi_1, \cdots, \pi_n; \pi_1', \cdots, \pi_{n-1}') = \frac{\partial \Omega}{\partial \omega_n}(\pi_1, \cdots, \pi_n; \omega_n; \pi_1', \cdots, \pi_{n-1}'),$$

which will be called Whittaker's function.

Differentiating the equation of energy with respect to π_m and π_p' $(m = 1, \cdots, n; p = 1, \cdots, n-1)$, and recalling that ω_n depends on π_m: and π_p' by (16), one obtains

(18) $$\frac{\partial \Omega}{\partial \pi_p'} = \omega_n \left(\frac{\partial^2 \Omega}{\partial \omega_n \partial \pi_p'} + \frac{\partial^2 \Omega}{\partial \omega_n^2} \frac{\partial \omega_n}{\partial \pi_p'} \right),$$

(19) $$\frac{\partial \Omega}{\partial \pi_m} = \omega_n \left(\frac{\partial^2 \Omega}{\partial \omega_n \partial \pi_m} + \frac{\partial^2 \Omega}{\partial \omega_n^2} \frac{\partial \omega_n}{\partial \pi_m} \right),$$
$$(m = 1, \ldots, n;\ p = 1, \ldots, n-1),$$

while from (17) we have

(20) $$\frac{\partial L'}{\partial \pi_m} = \frac{\partial^2 \Omega}{\partial \omega_n \partial \pi_m} + \frac{\partial^2 \Omega}{\partial \omega_n^2} \frac{\partial \omega_n}{\partial \pi_m},$$

(21) $$\frac{\partial L'}{\partial \pi_p'} = \frac{\partial^2 \Omega}{\partial \omega_n \partial \pi_p'} + \frac{\partial^2 \Omega}{\partial \omega_n^2} \frac{\partial \omega_n}{\partial \pi_p'},$$
$$(m = 1, \cdots, n;\ p = 1, \cdots, n-1).$$

Comparing equations (19) and (20), we have

$$\frac{\partial \Omega}{\partial \pi_m} = \omega_n \frac{\partial L'}{\partial \pi_m} , \tag{22}$$

and comparing equations (18) and (21), we have

$$\frac{\partial \Omega}{\partial \pi'_p} = \omega_n \frac{\partial L'}{\partial \pi'_p} , \tag{23}$$

Combining these results with equations (12) and (13), one obtains

$$\frac{\partial L}{\partial \omega_p} = \frac{\partial L'}{\partial \pi'_p} \quad , \quad \frac{\partial L}{\partial \pi_m} = \omega_n \frac{\partial L'}{\partial \pi_m}, \quad (p = 1, \cdots, n-1 \,;\, m = 1, \cdots, n). \tag{24}$$

Substituting these equations into the Hamel–Boltzmann equations of motion (2) ($P^*_s = 0$) and relations (5), and taking into account (9) and that $d/dt = \omega_n \, d/d\pi_n$, we obtain the system

$$\frac{d}{d\pi_m} \frac{\partial L'}{\partial \pi'_p} + \sum_{r=1}^{n-1} \sum_{t=l+1}^{n-1} \gamma^r_{tp} \frac{\partial L'}{\partial \pi'_r} \pi'_t - \frac{\partial L'}{\partial \pi_p} = Q^*_p , \qquad (p = l+1, \cdots, n-1), \tag{25}$$

$$\frac{dq_s}{d\pi_n} = \sum_{p=l+1}^{n-1} b_{sp} \pi'_p + b_{sn} , \qquad (s = 1, \cdots, n), \tag{26}$$

where

$$Q^*_p(\pi_m , \pi'_r) = - \sum_{r=1}^{n-1} \gamma^r_{np} \frac{\partial L'}{\partial \pi'_r} - \Pi_n \left(\gamma^n_{np} + \sum_{t=l+1}^{n-1} \gamma^n_{tp} \pi'_t \right) , \tag{27}$$

$$(m = 1, \cdots, n \,;\, r = l+1, \cdots, n-1).$$

Here, Π_n denotes the function $\partial L/\partial \omega_n$ where $\omega_1, \cdots, \omega_n$ are replaced by $\pi'_1, \ldots, \pi'_{n-1}$ using (9) and (16), i.e.

$$\Pi_n(\pi_1, \cdots, \pi_n \,;\, \pi'_1, \cdots, \pi'_{n-1}) = \frac{\partial L}{\partial \omega_n}(\pi_1, \cdots, \pi_n \,;\, \omega_1, \cdots, \omega_n). \tag{28}$$

Now these equations (25) may be regarded as the equations of motion of a new non-holonomic mechanical system where L' is Lagrange's function, π_n plays the role of time as the independent variable and the Q^*_p are "non-conservative forces". Thus the energy integral (8) enables us to reduce a given non-holonomic mechanical system with $(n - l)$ degrees of freedom to another non-holonomic mechanical system with $(n - 1 - l)$ degrees of freedom. When the mechanical system is holonomic and the quasi-coordinates are true coordinates and the equations (25) reduce to the known Whittaker equations (see [2], p. 66). Hence, the equations (25) may be called Whittaker's form of the Hamel-Boltzamann equations for non-holonomic conservative mechanical systems. As a rule when one uses the Hamel-Boltzmann equations (see for example [5], p. 371), one may omit in the function L' the nonlinear terms with respect to $\pi'_1, \cdots, \pi'_l$. Also, the condition that $\pi'_1 = \cdots = \pi'_l = 0$ can be used only after calculating the terms $\partial L'/\partial \pi'_r$.

Here we have $(2n - l - 1)$ equations (equations (25–26)) of the first order with respect to the $q_1, \cdots, q_n$; $\pi'_{l+1}, \cdots, \pi'_{n-1}$ whose solution yields these $(2n - l - 1)$ quantities as function of π_n

$$\pi'_p = \pi'_p(\pi_n, h, c_1, \cdots, c_{2n-l-1}), \qquad (p = l+1, \cdots, n-1), \tag{29}$$

$$q_s = q_s(\pi_n, h, c_1, \cdots, c_{2n-l-1}), \qquad (s = 1, \cdots, n), \tag{30}$$

where $h, c_1, \cdots, c_{2n-l-1}$ are constants of integration. When (29–30) are substituted in equation (16) and after integration one obtains a functional dependence between π_n and the time

$$t = \int \frac{d\pi_n}{\omega_n(\pi_n, h, c_1, \cdots, c_{2n-l-1})} + c_{2n-l}, \tag{31}$$

where c_{2n-l} is an integration constant. Hence, we have the complete solution of the equation of motion.

Combining (14), (17) and (9) we have Whittaker's function in the form

$$L' = \sum_{r=1}^{n-1} \frac{\partial L}{\partial \omega_r} \pi'_r + \frac{\partial L}{\partial \omega_n}, \tag{32}$$

where on the right hand side $\omega_1, \ldots, \omega_n$ must be eliminated using (9) and (16).

3. EXAMPLE

Let us illustrate the application of the present theory to a simple non-holonomic problem (see [4], p. 20). Two material particles M_1 and M_2 of unit mass $(m = 1)$ are joined by a rod of constant length b and negligible mass. The system can be moved only in the vertical plane and only in such a way that the velocity of the rod's centre is directed along the rod.

Let be the independent coordinates q_1, q_2, q_3 $(n = 3)$, where q_1 and q_2 are the Cartesian coordinates of the rod's centre (the axis Oq_2 is vertical) and q_3 is the angle between the rod and the Oq_1 axis.

Now, the Lagrangian function is

$$L = \dot{q}_1^2 + \dot{q}_2^2 + \frac{b^2}{4} \dot{q}_3^2 - 2 g q_2 \tag{33}$$

and the axial velocity of the rod's centre yields the non-holonomic constraint

$$-\dot{q}_1 \sin q_3 + \dot{q}_2 \cos q_3 = 0, \qquad (l = 1), \tag{34}$$

where g is the gravitational constant.

Let us introduce the quasi-velocities

$$\omega_1 = \dot{\pi}_1 = -\dot{q}_1 \sin q_3 + \dot{q}_2 \cos q_3, \tag{35}$$

$$\omega_2 = \dot{\pi}_2 = -\dot{q}_1 \cos q_3 - \dot{q}_2 \sin q_3, \tag{36}$$

$$\omega_3 = \dot{\pi}_3 = \dot{q}_3. \tag{37}$$

Comparing (35) with (34) it is obvious that $\omega_1 = 0$. From (37) we have that $\pi_3 = q_3$. Using (35–37) Lagrange's function transforms into

$$\text{(38)} \qquad L = \omega_2^2 + \frac{b^2}{4}\omega_3^2 - 2\,g q_2 + \cdots .$$

Combining (38), (9) and (8) ($s = 2\,;\, n = 3$) we have the energy integral in the form

$$\text{(39)} \qquad \omega_3^2\left(\frac{b^2}{4} + \pi_2'^2\right) = h - 2\,g q_2 \,,$$

whose solution is,

$$\text{(40)} \qquad \omega_3^2 = \frac{h - 2\,g q_2}{\dfrac{b^2}{4} + \pi_2'^2} \,.$$

Now, from (32), (38), (9) and (40) one obtains Whittaker's function

$$\text{(41)} \qquad L' = 2\sqrt{(h - 2\,g q_2)\left(\frac{b^2}{4} + \pi_2'^2\right)} \,.$$

In this example the objects of anholonomity takes the values

$$\text{(42)} \qquad \gamma_{12}^1 = \gamma_{13}^1 = \gamma_{12}^2 = \gamma_{23}^2 = 0 \quad ; \quad \gamma_{23}^1 = 1 \quad ; \quad \gamma_{13}^2 = -1\,;$$

$$\gamma_{\mu\mu}^{\nu} = 0 \quad ; \quad \gamma_{\tau\mu}^{\nu} = -\gamma_{\mu\tau}^{\nu} \quad ; \quad \gamma_{\mu\tau}^3 = 0\,; \qquad (\nu\,,\mu\,,\tau = 1\,,2\,,3).$$

Substituting (41–42) into (25) ($p = 2$) and using (7), (5) and (35–37) we obtain Whittaker's equation of motion (here is $n - 1 - l = 1$)

$$\text{(43)} \qquad (h - 2\,g q_2)\frac{d\pi_2'}{d\pi_3} - g \sin q_3\left(\frac{b^2}{4} + \pi_2'^2\right) = 0\,.$$

This equation must be solved together with the following equations

$$\text{(44)} \qquad \frac{dq_1}{d\pi_3} = -\cos q_3\, \pi_2' \,,$$

$$\text{(45)} \qquad \frac{dq_2}{d\pi_3} = -\sin q_3\, \pi_2' \,,$$

$$\text{(46)} \qquad \frac{dq_3}{d\pi_3} = 1 \,,$$

which are obtained by solving (35–37) with respect to $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$ and using (9) ($p = 2\,;\, n = 3$) and the fact that $\omega_1 = 0$.

The solutions to equations (43–46) are

$$\text{(47)} \qquad \pi_2' = -g c_1^2 \cos \pi_3 + c_2 \,,$$

$$\text{(48)} \qquad q_1 = -c_2 \sin \pi_3 + \frac{g c_1^2}{2}\pi_3 + \frac{g c_1^2}{4}\sin 2\,\pi_3 + c_3 \,,$$

$$\text{(49)} \qquad q_2 = c_2 \cos \pi_3 - \frac{g c_1^2}{2}\cos^2 \pi_3 - \frac{c_2^2}{2\,g c_1^2} + \frac{h}{2\,g} - \frac{b^2}{8\,g c_1^2} \,,$$

$$\text{(50)} \qquad q_3 = \pi_3 \,,$$

where h, c_1, c_2 and c_3 are constants of integration.

Substituting (47–50) into (40) and after integration ($\omega_3 = d\pi_3/dt$) we have the quasi-coordinate π_3 as a function of the time

(51) $$\pi_3 = \frac{t}{c_1} + c_4 ,$$

where c_4 is a constant of integration. Hence, we have the complete solution of the problem. This solution (equations (48–51)) is same as those obtained in [4], p. 53 by application of Appell's equations of motion for non-holonomic mechanical systems.

Acknowledgement. The author is grateful to Dr. A.M. Strauss for his constructive review of the manuscript.

REFERENCES

[1] E. WHITTAKER, *On the Reduction of the Order of the Differential Equations of a Dynamical Problem*, by «Use of the Integral of Energy, Messenger of Mathematics, Cambridge», 30, 93–97 (1900).

[2] E. WHITTAKER, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambr. Univer. Press, fourth ed. (1970)

[3] YU I. NEIMARK and N. A. FUFAEV, *Dynamics of the Non-Holonomic Systems*, « Izd. Nauka, F. », M. Lit. Moscow (1967) (in Russian).

[4] F. R. GANTMACHER, *Lectures on the Analytical Mechanics, Gosud*, « Izd. F. », M. Lit. Moscow (1960) (in russian).

[5] A. I. LURJE, *Analytical Mechanics*, «Gosud. Izd. F.», M. Lit. Moscow (1961) (in Russian).