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Some remarks on a class of Semi-Groups of Operators. II

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Topologia. — *Some remarks on a class of Semi-Groups of Operators. II.* Nota di VASILE I. ISTRĂŢESCU e ANA I. ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota sono dimostrati nuovi risultati sull'invertibilità dei semigruppi di operatori (T_t, $t \geq 0$) su spazi di Banach.

0. In [1, 3] some results are given about the class of all semi-groups of operators (T_t, $t \geq 0$) on a Banach space which have the property that for some $t > 0$, $T_t - I$ is compact or is an α–contraction or Riesz operators.

An important class of operators, generalizing the class of compact operators with important applications to Marcov processes is the class of quasi-compact operators [5]. Our purpose in the present Note is to obtain some results about semi–groups such that for some $t > 0$, $T_t - I$ is in a class of operators containing the class of quasi-compact operators.

1. First we recall some definitions for the reader's convenience.

Let X be a Banach space and A be a bounded set in X. We define $\alpha(A)$, the Kuratowski number of the set A, as the infimum of all $\varepsilon > 0$ such that there exists a decomposition of A into a finite number of subsets of diameter less than ε. It is easy to see that the following assertions hold:

1) if A, B are bounded set in X, then $A \subseteq B$ implies $\alpha(A) < \alpha(B)$;

2) if A is relatively compact, then $\alpha(A) = 0$;

3) if $A + B = \{a + b , a \in A , b \in B\}$ then $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

DEFINITION 1.1. *An operator* T *on* X *which is continuous is called quasi-compact if there exists an integer* m *and a compact operator* Q *such that*

$$\| T^m - Q \| < 1 .$$

DEFINITION 1.2. [4] *A function* $f : X \to X$ *is called locally* α*–contraction if for each bounded set* $A \subset X$, *there exists an integer* $n = n(A)$ *such that, if* $\alpha(A) > 0$,

$$\alpha(f^n A) \leq k\alpha(A)$$

where $k \in [0 , 1)$ *and is independent of* A. *For spectral properties of such class see* [2].

It is clear that if $n(A) = 1$ for all A then our class reduces to the class of α–contractions introduced by Darbo. It is easy to see that the following assertion holds.

(*) Nella seduta del 12 gennaio 1974.

THEOREM 1.3. *Every quasi-compact operator on the Banach space* X *is locally α–contraction.*

Proof. We can take $k = \| T^m - Q \|$ and $n(A) = m$ for all bounded sets in X.

This result motivates our considerations.

2. Let $\{T_t\}_{t \geq 0}$ be a semi-group of class C_0 on X and define the set C_{loc} as

$$\{t, t > 0, T_t - I \text{ is locally } \alpha\text{–contraction}\}$$

It is clear that these semi-groups contain the semi-groups studied in the papers [1] and [3].

Also we suppose that the semi-group $\{T_t\}_{t \geq 0}$ satisfies $\| T_t \| \leq Me^{wt}$ for all $t \geq 0$ with constants M and w.

THEOREM 2.1. *If* $C_{loc} \neq \varnothing$ *then* T_t *is invertible for all t.*

Proof. Suppose that the assertion is false. Then $0 \in \sigma(T_t)$ for some $t > 0$ and thus for all $t > 0$. Since $T_t - I$ is locally α–contraction we have that the subspace $N(T_t)$ is finite dimensional. Indeed, if this is not so, let $\mathcal{A} = \{x, x \in N(T_t), \|x\| = 1\}$ then we have, if $\alpha(A) < 0$

$$\alpha((T_t - I)^{n(A)} A) = \alpha(A) < \alpha(A)$$

and the assertion is proved. We can show that $0 \in \sigma_p(T_t)$, the point spectrum of T_t. Indeed, we find a sequence of unit vectors $\{x_n\}$ such that $Tx_n \to 0$ and we wish to show that $\{x_n\}$ is relatively compact. If $m = n(\{x_n\})$ then clear

$$\alpha(\{x_n\}) = \alpha((T_t - I)^m \{x_n\}) < \alpha(\{x_n\})$$

since for all $i \in [1, n]$, $T^i x_n \to 0$ and thus $\alpha(\{T^i x_n\}) = 0$. From this clear $0 \in \sigma_p(T_t)$.

We can now follow the arguments in [1] to obtain the assertion of the theorem.

The following theorem represents a more direct extension of the result in [1] about the invertibility of T_t.

Let $\{T_t\}$ be a semi-group of class C_0 on X and define the set C_R as

$$\{t, t > 0, T_t - I \text{ is of Riesz type}\}.$$

Since every compact operator is of Riesz type clear this set is larger than the set C defined in [1].

We have the following results stated in [3] Theorem 2.2 but the proof was not explicitly indicated.

THEOREM 2.2. *If* $C_R \neq \varnothing$ *then* T_t *is invertible for all t.*

Proof. If the assertion is false, $0 \in \sigma(T_t)$ for some, and then for all $t > 0$. We wish to show that $N(T_t)$ is finite dimensional. Since $T_t - I$ is of Riesz type then, by Ruston's characterization of Riesz operators T, we have that, if

$$\lambda_m(T) = \inf \| T^m - C \|$$

where the infimum is taken over all compact operators.

$$\lim \lambda_m^{1/m}(T) = 0$$

and from this clear $T_t - I$ is a quasi-compact operator. Our assertion follows now from Theorem 2.1.

The proof of the theorem can be continued as in the paper [1].

Remark. Consider the measure of noncompactness $\alpha(\cdot)$ of Kuratowski and for each bounded operator T in the space $\mathfrak{L}(X)$ define

$$T \to p(T) = \inf \{ k > 0, \text{T is a } k\text{-contraction} \}$$

and it is easy to see that $p(\cdot)$ is seminorm on $\mathfrak{L}(X)$ ($p(T) = 0$ if and only if T is compact). It is easy to see that Theorem 2.2 is valid if $T_t - I$ is supposed to be with the property

$$\underline{\lim}\, p(T_t - I)^{1/m} = 0 .$$

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