Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Hary M. Srivastava, Harold Exton

A transformation formula relating two Lauricella functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **56** (1974), n.1, p. 38–42.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1974_8_56_1_38_0>

Funzioni speciali. — *A transformation formula relating two Lauricella functions* (*). Nota di HARY M. SRIVASTAVA (**) e HAROLD EXTON, presentata (***) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori dimostrano che le funzioni $F_A^{(n)}$, $F_C^{(n)}$ di *Lauricella* in n variabili sono legate linearmente. I casi $n = 1$, $n = 2$ erano noti; il caso $n \geqq 3$ è nuovo.

1. INTRODUCTION

In the theory of hypergeometric functions it is fairly well known that Lauricella [5] generalized the Appell double hypergeometric functions $F_1, \cdots, F_4$ (cf., e.g., [3], p. 224) to functions of n variables. Two of Lauricella functions, viz. $F_A^{(n)}$ and $F_C^{(n)}$, which indeed are generalizations of Appell's F_2 and F_4, respectively, are defined by means of the multiple series [5, p. 113]

$$F_A^{(n)}[\alpha, \beta_1, \cdots, \beta_n; \gamma_1, \cdots, \gamma_n; x_1, \cdots, x_n] = \tag{1}$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$|x_1| + \cdots + |x_n| < 1;$$

and

$$F_C^{(n)}[\alpha, \beta; \gamma_1, \cdots, \gamma_n; x_1, \cdots, x_n] = \tag{2}$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$|x_1|^{1/2} + \cdots + |x_n|^{1/2} < 1;$$

where, as usual,

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } m = 0, \\ \lambda(\lambda+1)\cdots(\lambda+m-1), & \text{if } m = 1, 2, 3, \cdots. \end{cases} \tag{3}$$

The object of the present Note is to show how these two Lauricella functions are connected linearly. Indeed we establish the following transformation

(*) For an abstract of this paper see « Notices Amer. Math. Soc. », *21* (1974), p. A-6.

(**) The work of this author was supported in part by NRC grant A-7353.

(***) Nella seduta del 12 gennaio 1974.

formula:

$$F_C^{(n)}\left[\frac{1}{2}\alpha\,,\,\frac{1}{2}\alpha+\frac{1}{2}\,;\,\gamma_1,\cdots,\gamma_n\,;\,x_1^2,\cdots,x_n^2\right]= \tag{4}$$

$$=(1+x_1+\cdots+x_n)^{-\alpha}\,F_A^{(n)}\Big[\alpha\,,\,\gamma_1-\frac{1}{2}\,,\cdots,\gamma_n-\frac{1}{2}\,;$$

$$2\gamma_1-1\,,\cdots,2\gamma_n-1\,;\,\frac{2x_1}{1+x_1+\cdots+x_n}\,,\cdots,\frac{2x_n}{1+x_1+\cdots+x_n}\Big],$$

provided

$$\sum_{i=1}^{n}\left|\frac{x_i}{1+x_1+\cdots+x_n}\right|<\frac{1}{2}\,. \tag{5}$$

By a simple change of variables this last formula (4) can be put in its equivalent form:

$$F_A^{(n)}\left[\alpha\,,\,\gamma_1-\frac{1}{2}\,,\cdots,\gamma_n-\frac{1}{2}\,;\,2\gamma_1-1\,,\cdots,2\gamma_n-1\,;\,2x_1\,,\cdots,2x_n\right]= \tag{6}$$

$$=(1-x_1-\cdots-x_n)^{-\alpha}\,F_C^{(n)}\Big[\frac{1}{2}\alpha\,,\,\frac{1}{2}\alpha+\frac{1}{2}\,;\,\gamma_1,\cdots,\gamma_n\,;$$

$$\frac{x_1^2}{(1-x_1-\cdots-x_n)^2}\,,\cdots,\frac{x_n^2}{(1-x_1-\cdots-x_n)^2}\Big],$$

which holds true when $|x_1|+\cdots+|x_n|<1/2$.

2. Derivation of formula (4)

From Euler's definition of the Γ- function, viz. (cf., e.g., [3], p. 1)

$$\Gamma(z)=\int_0^\infty e^{-t}\,t^{z-1}\,dt\quad,\quad Re(z)>0\,, \tag{7}$$

it readily follows that [7, p. 101]

$$(\lambda)_m=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}\,t^{\lambda+m-1}\,dt\,, \tag{8}$$

where $(\lambda)_m$ is defined by (3), $m=0\,,1\,,2\,,\cdots$, and $Re(\lambda)>0$.

Making use of (8) and Legendre's duplication formula [3, p. 5 (15)] in the more convenient form

$$\left(\frac{1}{2}\lambda\right)_m\left(\frac{1}{2}\lambda+\frac{1}{2}\right)_m=2^{-2m}\,(\lambda)_{2m}\,,\qquad m=0\,,1\,,2\,,\cdots, \tag{9}$$

in conjunction with the definition (2), we arrive fairly easily at the integral representation

$$\text{(10)} \qquad F_C^{(n)}\left[\frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2}; \gamma_1, \cdots, \gamma_n; x_1^2, \cdots, x_n^2\right]$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} {}_0F_1\left[-; \gamma_1; \frac{x_1^2 t^2}{4}\right] \cdots {}_0F_1\left[-; \gamma_n; \frac{x_n^2 t^2}{4}\right] dt,$$

provided, by analytic continuation, that $Re(\alpha) > 0$ and

$$\text{(11)} \qquad |x_1| + \cdots + |x_n| < 1.$$

Formula (10) does not appear to have been noticed earlier.

Now we apply Kummer's second theorem [6, p. 12]

$$\text{(12)} \qquad {}_0F_1\left[-; \lambda; \frac{x^2}{4}\right] = e^{-x} {}_1F_1\left[\lambda - \frac{1}{2}; 2\lambda - 1; 2x\right]$$

to each of the hypergeometric ${}_0F_1$ functions on the right-band side of formula (10), expand the integrand in powers of t, and change the order of integration and summations, which can readily be justified. On evaluating the resulting integral by using (7), and on making an appeal to the principle of analytic continuation, we shall obtain our transformation formula (4) under condition (5).

3. Alternative proofs

For a direct proof of (4) we note an immediate consequence of Kummer's theorem (12) in the form

$$\text{(13)} \qquad {}_0F_1\left[-; \gamma_1; \frac{x_1^2}{4}\right] \cdots {}_0F_1\left[-; \gamma_n; \frac{x_n^2}{4}\right] =$$

$$= e^{-(x_1+\cdots+x_n)} {}_1F_1\left[\gamma_1 - \frac{1}{2}; 2\gamma_1 - 1; 2x_1\right] \cdots {}_1F_1\left[\gamma_n - \frac{1}{2}; 2\gamma_n - 1; 2x_n\right].$$

Now replace x_i by $x_i t$, $i = 1, \cdots, n$, multiply both sides by $t^{\alpha-1}$, take their Laplace transforms using the elementary integral (7), and the transformation formula (4) would follow at once.

Alternatively, the equivalent formula (6) can be proven fairly simply by using Gauss's summation theorem [3, p. 104]

$$\text{(14)} \qquad {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad Re(c-a-b) > 0,$$

$$c \neq 0, -1, -2, \cdots.$$

Indeed, the coefficient of $x_1^{m_1} \cdots x_n^{m_n}$ on the right-hand side of formula (6) is given by $\Delta(m_1, \cdots, m_n)$, where

$$\frac{m_1! \cdots m_n!}{(\alpha)_{m_1+\cdots+m_n}} \Delta(m_1, \cdots, m_n) =$$

$$= \sum_{k_1=0}^{[m_1/2]} \cdots \sum_{k_n=0}^{[m_n/2]} \frac{\left(\frac{1}{2}\alpha\right)_{k_1+\cdots+k_n} \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_{k_1+\cdots+k_n} (-m_1)_{2k_1} \cdots (-m_n)_{2k_n}}{k_1! \cdots k_n! \, (\alpha)_{2k_1+\cdots+2k_n} (\gamma_1)_{k_1} \cdots (\gamma_n)_{k_n}} =$$

$$= {}_2F_1\left[-\frac{1}{2}m_1, -\frac{1}{2}m_1+\frac{1}{2}; \gamma_1; 1\right] \cdots {}_2F_1\left[-\frac{1}{2}m_n, -\frac{1}{2}m_n+\frac{1}{2}; \gamma_n; 1\right],$$

by using the duplication formula (9), and if we apply Gauss's theorem (14) to sum each of these hypergeometric series, we shall finally get

$$(15) \qquad \Delta(m_1, \cdots, m_n) = \frac{2^{m_1+\cdots+m_n} (\alpha)_{m_1+\cdots+m_n} \left(\gamma_1-\frac{1}{2}\right)_{m_1} \cdots \left(\gamma_n-\frac{1}{2}\right)_{m_n}}{m_1! \cdots m_n! \, (2\gamma_1-1)_{m_1} \cdots (2\gamma_n-1)_{m_n}},$$

for integers $m_i \geqq 0$, $i = 1, \cdots, n$.

This evidently completes the proof of (6) using only the summation theorem (14).

4. PARTICULAR CASES

For $n = 1$, formula (4) yields the quadratic transformation

$$(16) \qquad {}_2F_1\left[\frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2}; \gamma; x^2\right] = (1+x)^{-\alpha} {}_2F_1\left[\alpha, \gamma-\frac{1}{2}; 2\gamma-1; \frac{2x}{1+x}\right],$$

valid when $|x/(1+x)| < 1/2$. By setting $x = z/(2-z)$ and $\gamma = \beta + 1/2$, this formula would reduce to its equivalent form

$$(17) \qquad {}_2F_1[\alpha, \beta; 2\beta; z] =$$

$$= \left(1-\frac{1}{2}z\right)^{-\alpha} {}_2F_1\left[\frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2}; \beta+\frac{1}{2}; \frac{z^2}{(2-z)^2}\right], \quad |z| < 1.$$

which was given by Goursat (1) ([4], p. 120, Eq. (45); see also [3], p. 111, Eq. (4)).

On the other hand, a special case of formula (6) when $n = 2$ leads to Bailey's transformation [1, p. 11, Eq. (3.1)]

$$(18) \qquad F_2\left[\alpha, \gamma-\frac{1}{2}, \gamma'-\frac{1}{2}; 2\gamma-1, 2\gamma'-1; 2x, 2y\right] =$$

$$= (1-x-y)^{-\alpha} F_4\left[\frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2}; \gamma, \gamma'; \frac{x^2}{(1-x-y)^2}, \frac{y^2}{(1-x-y)^2}\right],$$

$$|x| + |y| < \frac{1}{2},$$

(1) It may be of interest to remark that formula (17) is essentially the same as the hypergeometric transformation (52), p. 78 of E. E. Kummer [« J. Reine, Angew. Math. », *15*, 39-83 (1836)].

associated with Appell functions of two variables of the second and fourth types (cf. [3], p. 224).

It may be of interest to remark that Bailey's derivation of the quadratic transformation (18) runs parallel to our alternative proof, given in the preceding section, of the generalization (6). We shall demonstrate here that formula (18) is a fairly immediate consequence of the known transformation (17).

Indeed, by rewriting the double series of Appell's F_2 function as

$$(19)\qquad F_2[\alpha,\beta,\beta';\gamma,\gamma';x,y]=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\,{}_2F_1[\alpha+n,\beta;\gamma;x]\frac{y^n}{n!},$$

and applying (17) to each term, we obtain the transformation

$$(20)\qquad F_2[\alpha,\beta,\beta';2\beta,\gamma';2x,y]$$
$$=(1-x)^{-\alpha}H_4\left[\alpha,\beta';\beta+\frac{1}{2},\gamma';\frac{x^2}{4(1-x)^2},\frac{y}{1-x}\right],$$

which is equivalent to an earlier result proved in a different way by Erdélyi [2, p. 381, Eq. (7.4)], H_4 being one of Horn functions defined by [3, p. 225]

$$(21)\qquad H_4[\alpha,\beta;\gamma,\gamma';x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

Another application of (17) to the second member of (20), with $\gamma'=2\beta'$, would finally yield Bailey's formula (18), since the definition (21) in conjunction with (17) leads us immediately to the quadratic transformation

$$(22)\qquad H_4[\alpha,\beta;\gamma,2\beta;x,2y]=$$
$$=(1-y)^{-\alpha}F_4\left[\frac{1}{2}\alpha,\frac{1}{2}\alpha+\frac{1}{2};\gamma,\beta+\frac{1}{2};\frac{4x}{(1-y)^2},\frac{y^2}{(1-y)^2}\right].$$

This last formula (22) was derived earlier by Erdélyi [2, p. 382, Eq. (7.6)] by transforming the Pochhammer double-loop contour integral representing Appell's F_2 function.

References

[1] W. N. Bailey, *The generating function of Jacobi polynomials*, « J. London Math. Soc. », *13*, 8–12 (1938).

[2] A. Erdélyi, *Transformations of hypergeometric functions of two variables*, « Proc. Roy. Soc. Edinburgh Sect. A. », *62*, 378–385 (1948).

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. I, McGraw–Hill Book Co., Inc., New York–Toronto–London, 1953.

[4] E. Goursat, *Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique*, « Ann. Sci. École Norm. Sup. (2) », *10*, 3–142 (1881).

[5] G. Lauricella, *Sulle funzioni ipergeometriche a più variabili*, « Rend. Circ. Mat. Palermo », *7*, 111–158 (1893).

[6] L. J. Slater, *Confluent hypergeometric functions*, University Press, Cambridge, 1960.

[7] H. M. Srivastava, *Some integrals representing triple hypergeometric functions*, « Rend. Circ. Mat. Palermo ser. II », *16*, 99–115 (1967).