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Asymptotically Self-invariant Sets and Random Functional Differential Systems

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Sistemi differenziali aleatori. — *Asymptotically Self-invariant Sets and Random Functional Differential Systems.* Nota di G. S. LADDE, V. LAKSHMIKANTHAM e PAN-TAI LIU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota gli Autori studiano il comportamento di un insieme asintoticamente autoinvariante rispetto ad un sistema di equazioni funzionali differenziali aleatorie.

1. INTRODUCTION

The concept of an asymptotically self-invariant set was introduced in [3]. In [4] and [5], the stability behavior of an asymptotically self-invariant set with respect to nonlinear systems under constant perturbations was investigated. In the present paper, we wish to do the same thing for a system described by stochastic functional differential equations. Our approach again depends on the construction of Lyapunovlike functional (section 3). The main results are given in section 4.

2. DEFINITIONS AND MOTIVATION FOR THE MAIN RESULT

Consider the random functional differential system

$$x'(t) = f(t, x_t, \omega) \qquad x_{t_0}(\omega) = \varphi_0(\omega), \tag{2.1}$$

where $f \in C[R^+ \times C_\rho, R^n]$, where

$$C_\rho = [\varphi \in C^n : \|\varphi\|_0 < \rho].$$

Here $C^n = C[[-\tau, 0], R^n]$ and $\|\varphi\|_0 = \max\limits_{-\tau \leq s \leq 0} \|\varphi(s)\|$, $\|\cdot\|$ being any norm in R^n and τ is a random delay defined on a complete probability space $(\Omega, \mathcal{F}, p)$ with values in $[0, h]$. We follow the standard notation for functional differential equations [3]. We shall assume, throughout this paper, that (2.1) has a least one solution $\{x_t(t_0, \varphi_0), t < t_0\}$, which is a measurable and separable stochastic process.

DEFINITION 2.1. *A function* $\lambda \in C[R^+, R^+]$ *is said to belong to class* $\mathfrak{L}$ *if* $\lambda(t)$ *is decreasing in* t *and* $\lim\limits_{t\to\infty} \lambda(t) = 0$.

DEFINITION 2.2. *The random set* $\varphi = 0$ *is said to be asymptotically self-invariant* (ASI) *with respect to system* (2.1) *if*

$$\|x_t(t_0, 0)\|_0 \leq q(t_0), \qquad \text{w.p.1.}, \qquad t \geq t_0, \tag{2.2}$$

where $q \in \mathfrak{L}$ *and* $x_t(t_0, 0)$ *is any solution of* (2.1).

(*) Nella seduta del 12 gennaio 1974.

For an equivalent definition of ASI sets and their stability properties, see [3, 4, 5].

We shall assume that the random set $\varphi = 0$ is ASI with respect to the system (2.1)

DEFINITION 2.3. *The* ASI *random set* $\varphi = 0$ *of* (2.1) *is said to be*:

(i) *Almost sure equi-stable, if for each* $t_0 \in R^+$

$$\| x_t(t_0, \varphi_0) \| \leq K(t_0, \tau) \| \varphi \|_0 + q(t_0), \quad \text{w.p.l.}, \quad t \geq t_0,$$

where $K \in C[R^+ \times R^+, R^+]$ *and* $q \in \mathfrak{L}$;

(ii) *Almost sure equi-exponentially asymptotically stable, if*

$$\| x_t(t_0, \varphi_0) \|_0 \leq K(t_0, \tau) \| \varphi_0 \| e^{-\alpha(t,t_0)} + H(t, t_0), \quad \text{w.p.l.}, \quad t \geq t_0, \tag{2.3}$$

where $K \in C[R^+ \times R^+, R^+]$, $\alpha > 0$, $H \in C[R^+ \times R^+, R^+]$, $H(t, t) = 0$, $H(t, t_0) \leq p(t_0)$, $p \in \mathfrak{L}$ *and* $\lim\limits_{t \to \infty} [\sup\limits_{t_0 \geq T_0} H(t, t_0)] = 0$ *for some positive number* $T_0 > 0$.

Obviously (ii) implies (i), furthermore, if $q(t_0) \equiv 0$ and if $H(t, t_0) \equiv 0$, then the Definition 2.3 reduces to almost sure equi-stability and almost sure equi-exponential asymptotic stability of the trivial solution of (2.1) [2]. For the Definition 2.3, it is not necessary to assume that $f(t, 0) \equiv 0$, w.p.l.

In the following, we will discuss an example to illustrate the meaning of the Definition 2.3 and to give the motivation of our main result in the subsequent sections. Consider the linear system with random delay

$$x'(t) = A(t) \times (t) + B(t) \times (t - \tau) + w(t) \tag{2.4}$$

with the initial function φ_0 at $t = t_0$, where $A(t)$, $B(t)$ are $n \times n$ continuous matrix functions and $w \in C[R^+, R^n]$. $\{w(t), t \geq t_0\}$ is an n–dimensional stochastic process which is measurable and $\int_{t_0}^{t} E \| w(t) \| dt < \infty$, τ is random delay.

Analogous to the ordinary functional differential system, by following Bellman and Cook [1], let us introduce the random matrix function $Y(s, t)$ having the initial conditions

$$Y(t, t) = I \text{ (unit matrix)},$$

$$Y(s, t) \equiv 0, \quad \text{for} \quad t < s \leq t + \tau;$$

and $Y(s, t)$, as a function of s, satisfies the random matrix equation

$$\frac{dY}{ds}(s, t) = - Y(s, t) A(s) - Y(s + \tau, t) B(s + \tau), \quad s < t.$$

We further assume as in [4], that

$$\| B(t) \| \leq B_0, \qquad t_0 \leq t \leq t_0 + h \tag{2.5}$$

$$\| Y(s, t) \| \leq N \exp [\beta s - \alpha(t - s)], \quad \text{w.p.l.}, \quad N > 1, \quad \alpha > 0, \quad \beta \geq 0.$$

By following the proof in [1] any solution of (2.4) satisfies the random integral equation

$$x(t_0, \varphi_0)(t) = Y(t_0, t)\, \varphi_0(0) + \int_{t_0-\tau}^{t_0} Y(s+\tau, t)\, B(s+\tau)\, \varphi_0(s)\, ds + \int_{t_0}^{t} Y(s, t)\, w(s)\, ds. \tag{2.6}$$

This, together with the foregoing assumptions, yields the estimate

$$\| x(t_0, \varphi_0)(t) \| \le N e^{\beta t_0} [1 + \tau B_0\, e^{(\alpha+\beta)\tau}]\, \| \varphi_0 \|_0\, e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)} N e^{\beta s} \| w(s) \|\, ds, \quad \text{w.p.1.}, \quad t \ge t_0 .$$

which further implies

$$\| x_t(t_0, \varphi_0) \|_0 \le K(t_0, \tau)\, \| \varphi_0 \|_0\, e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)}\, \gamma(s)\, ds, \quad \text{w.p.1.}, \tag{2.7}$$

where $K(t_0, \tau) = N e^{\beta t_0 + \alpha\tau} [1 + \tau B_0\, e^{(\alpha+\beta)\tau}]$ and $\gamma(t) = N e^{\alpha h}\, e^{\beta t}\, \| w(t) \|$.

Setting $H(t, t_0) = \int_{t_0}^{t} e^{-\alpha(t-s)}\, \gamma(s)\, ds$ and assuming that $w(t)$ is such that

$$\int_{t}^{t+1} \gamma(s)\, ds \to 0 \quad \text{as} \quad t \to \infty. \tag{2.8}$$

which follows easily that $\lim_{t\to\infty} [\sup_{t_0 \ge 1} H(t, t_0)] = 0$ and $H(t, t_0) \le p(t_0)$, $t \ge t_0$, $p \in \mathfrak{L}$. For the proof of this assertion, see [3, vol. I, pp. 113]. Thus, we can conclude from (2.7) the random set $\varphi = 0$ is ASI with respect to the linear differential system (2.4) with random delay and it is almost sure equi-exponentially asymptotically stable.

Let $x_t(t_0, \varphi_0)$, $x_t(t_0, \psi_0)$ be solutions of (2.4), then we can obtain

$$\| x_t(t_0, \varphi_0) - x_t(t_0, \psi_0) \|_0 \le K(t_0, \tau)\, \| \varphi_0 - \psi_0 \|_0\, e^{-\alpha(t-t_0)}, \quad \text{w.p.1.}, \quad t \ge t_0, \tag{2.9}$$

on the basis of the assumptions (2.5).

Thus we have proved the following statement.

LEMMA 2.1. *Let us assume that the conditions* (2.5) *and* (2.8) *are satisfied. Then, the random set* $\varphi = 0$ *is* ASI *relative to the linear differential systems with random delay* (2.4) *and it is almost sure exponentially asymptotically stable. Furthermore, for any two solutions* $x_t(t_0, \varphi_0)$, $x_t(t_0, \psi_0)$, *the relation* (2.9) *is valid.*

Let us now introduce the concept of almost sure extreme equi-exponential asymptotic stability.

DEFINITION 2.4. *The stochastic functional differential system* (2.1) *is said to be almost sure extremely equi-exponentially asymptotically stable, if*

$$(2.10) \qquad \| x_t(t_0, \varphi_0) - x_t(t_0, \psi_0) \|_0 \leq K(t_0, \tau) \| \varphi_0 - \psi_0 \| e^{-\alpha(t-t_0)}, \qquad t \geq t_0$$

with probability one, where $K \in C[R^+ \times R^+, R^+]$, $\alpha > 0$.

Remark 2.1. Assume that a solution $x_t(t_0, 0)$ of (2.1) satisfies the relation

$$(2.11) \qquad \| x_t(t_0, 0) \| \leq H(t, t_0), \quad \text{w.p.l.}, \quad t \geq t_0$$

$$H \in C[R^+ \times R^+, R^+] \quad , \quad H(t, t) = 0 \quad , \quad H(t, t_0) \leq p(t_0), \ p \in \mathfrak{L}$$

and $\lim_{t \to \infty} [\sup_{t_0 \geq T_0} H(t, t_0)] = 0$, for some positive number $T_0 > 0$.

This, together with the Definition 2.2, and the relation (2.10), one can obtain the following inequality with probability one

$$(2.12) \qquad \| x_t(t_0, \varphi_0) \|_0 \leq K(t_0, \tau) \| \varphi \|_0 e^{-\alpha(t-t_0)} + H(t, t_0), \qquad t \geq t_0,$$

which shows that the ASI random set $\varphi = 0$ almost sure equi-exponentially asymptotically stable. Furthermore, notice that the relation (2.10) implies the uniqueness of solutions of (2.1) w.p.l.

3. CONSTRUCTION OF LYAPUNOV-LIKE FUNCTIONAL

In this section, we shall construct a Lyapunov functional which will be utilized later on.

THEOREM 3.1. *Assume that*

(i) *the stochastic functional differential system* (2.1) *is almost sure extremely equi-exponentially asymptotically stable*;

(ii) *the solution* $x_t(t_0, 0)$ *of* (2.1) *satisfies the relation*

$$\| x_t(t_0, 0) \|_0 \leq H(t, t_0), \quad \text{w.p.l.}, \quad t \geq t_0,$$

$$H \in C[R^+ \times R^+, R^+] \quad , \quad H(t, t) = 0 \quad , \quad H(t, t_0) \leq p(t_0), \qquad p \in \mathfrak{L}$$

and $\lim_{t \to \infty} [\sup_{t_0 \geq T_0} H(t, t_0)] = 0$, *for some positive number* $T_0 > 0$ *and furthermore* $H(t, t_0)$ *is partially differentiable with respect to* t_0 *and*

$$(3.1) \qquad \sup_{\sigma \geq 0} - \frac{\partial H}{\partial t_0}(t + \sigma, t) e^{\alpha\sigma} \leq \eta(t),$$

where $\eta \in C[R^+, R^+]$.

Then there exists a functional $V(t, \varphi)$ *with the following properties*:

(*a*) $V \in C[R^+ \times C_\rho, R^+]$ *and* V *is Lipschitzian in* φ *for the random function* $K(t, \tau)$; *w.p.l.*;

(*b*) $\| \varphi \|_0 \leq V(t, \varphi) \leq K(t, \tau) \| \varphi \|_0$, w.p.l., $t \in R^+$, $\varphi \in C_\rho$;

(*c*) $$D^+ V(t, \varphi) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x_{t+h}(t, \varphi)) - V(t, \varphi)] \leq$$
$$\leq -\alpha V(t, \varphi) + \eta(t), \quad \text{w.p.l.}, \quad t \in R^+, \quad \varphi \in C_\rho.$$

Proof: Define*

$$V(t,\varphi)=\sup_{\sigma\geq 0}\,[\,\|x_{t+\sigma}(t,\varphi)\|_0-H(t+\sigma,t)]\,e^{\alpha\sigma}. \tag{3.2}$$

where $x_{t+\sigma}(t,\varphi)$ is the solution of (2.1) for $t\in R^+$, $\varphi\in C_\rho$, with probability one. Clearly, from (2.12), it follows that $V(t,\varphi)$ is defined on $R^+\times C_\rho$ and satisfies the property (*b*) w.p.l. Moreover, using the uniqueness of solutions as mentioned in the Remark 2.1 and the assumption (ii), we get successively

$$\begin{aligned}
D^+V(t,\varphi)\leq \lim_{h\to 0^+}\sup\frac{1}{h}\,[\sup_{\sigma\geq 0}\{\|x_{t+\sigma+h}(t+h\,,x_{t+h}(t,\varphi))\|_0\,-\\
-H(t+h+\sigma\,,t+h)\}\,e^{\alpha\sigma}-\sup_{\sigma\geq 0}\{\|x_{t+\sigma}(t,\varphi)\|_0\\
-H(t+\sigma\,,t)\}\,e^{\alpha\sigma}\}=
\end{aligned}$$

$$\begin{aligned}
&=\lim_{h\to 0^+}\sup_{\sigma\geq h}\frac{1}{h}\,[\,\sup\{\,\|x_{t+\sigma}(t,\varphi)\|_0-H(t+\sigma\,,t+h)\}\,e^{\alpha(\sigma-h)}\,-\\
&-\sup_{\sigma\geq 0}\,\{\|x_{t+\sigma}(t,\varphi)\|_0-H(t+\sigma\,,t)\}\,e^{\alpha\sigma}]\,-\\
&-\lim_{h\to 0^+}\sup\frac{1}{h}\,[\sup_{\sigma\geq 0}\{\,\|x_{t+0}(t,\varphi)\|_0-H(t+\sigma\,,t+h)\}\,e^{\alpha(\sigma-h)}\,-\\
&-\sup_{\sigma\geq 0}\{\,\|x_{t+\sigma}(t,\varphi)\|_0-H(t+\sigma\,,t)\}\,e^{\alpha\sigma}]\\
&\leq V(t,\varphi)\lim_{h\to 0^+}\sup\frac{1}{h}(e^{-\alpha h}-1)\,+\\
&+\sup_{\sigma\geq 0}\left[\lim_{h\to 0^+}\sup\frac{1}{h}\,\{H(t,\sigma,t)-H(t+\sigma\,,t+h)\}\,e^{\alpha(\sigma-h)}\leq\right.\\
&\leq-\alpha V(t,\varphi)+\left.\sup_{\sigma\geq 0}\left\{-\frac{\partial H}{\partial t_0}(t+\sigma\,,t)\right\}e^{\alpha\sigma}\right]\leq\\
&\leq-\alpha V(t,\varphi)+\eta(t)
\end{aligned}$$

which proves (*c*). To show $V(t,\varphi)$ is Lipschitzian in φ, let $\varphi,\psi\in C_\rho$. Then, using the estimate (2.10), we have

$$\begin{aligned}
|V(t,\varphi)-V(t,\psi)|&=|\sup_{\sigma\geq 0}\{\|x_{t+\sigma}(t,\varphi)\|_0-H(t+\sigma\,,t)\}\,e^{\alpha\sigma}\\
&-\sup_{\sigma\geq 0}\{\|x_{t+\sigma}(t,\psi)\|_0-H(t+\sigma\,,t)\}\,e^{\alpha\sigma}|\\
&\leq\sup_{\sigma\geq 0}\|x_{t+\sigma}(t,\varphi)-x_{t+\sigma}(t,\psi)\|_0\,e^{\alpha\sigma}\\
&\leq K(t,\tau)\,\|\varphi-\psi\|_0\,,\qquad \text{w.p.l.}
\end{aligned}$$

The continuity of $V(t,\varphi)$ may be proved as in Theorem 7.2.1 [3]. The proof is complete.

Remark 3.1. In the deterministic case, the converse theorem for the exponential asymptotic stability of the trivial solution is well known [3] and has been successfully employed in studying the behavior of solutions of perturbed systems [3]. Theorem 3.1 is a corresponding result establishing the existence of a Lyapunov functional when the ASI random set $\varphi = 0$ of (2.1) is almost sure equi-exponentially asymptotically stable. This generalization naturally is a powerful tool in discussing the almost sure equi-stability properties of the ASI random sets of random perturbed systems.

Remark 3.2. If $f(t, \varphi)$ is non-linear and satisfies a Lipschitz condition is φ for a constant $L > 0$, one might suspect that simply assuming almost sure equi-exponential asymptotic stability of the ASI random set $\varphi = 0$, the proof of Theorem 7.2.2 [3] may be adapted without the almost sure extreme equi-exponential asymptotic stability and the relation (2.11). This is not true, since the assumption that $K(t, \tau)$ is bounded w.p.l., is crucial for the proof of Theorem 7.2.2 [3] to work and in our case, this is not the situation. See, for example, the definition of $K(t, \tau)$ relative to the equation (2.4). This remark justifies the assumption that the system (2.1) is almost sure extremely equi-exponentially asymptotically stable, which is necessarily satisfied in the case of equation (2.4). Furthermore, as pointed out in the Remark 2.1 we need not assume uniqueness conditions on $f(t, \varphi)$. Hence, weakening the hypotheses of Theorem 7.2.2 [3].

Remark 3.3. We shall make a comment on the definition of $V(t, \varphi)$ as given in (3.2). Had we started with the functional

$$V(t, \varphi) = \sup_{\sigma \geq 0} \| x_{t+\sigma}(t, \varphi) \| e^{\alpha\sigma},$$

and assumed that $f(t, \varphi)$ is linear in φ, we could obtain, as in the proof of Theorem 7.2.1 in [3], all the properties of $V(t, \varphi)$ except the upper bound. For, in this case, we could obtain

$$V(t, \varphi) \leq K(t, \tau) \| \varphi \|_0 + \sup_{\sigma \geq 0} H(t + \sigma, t) e^{\alpha\sigma}, \qquad \text{w.p.l.}$$

and we are not all sure that $\sup_{\sigma \geq 0} H(t + \sigma, t) e^{\alpha\sigma}$ exists. As an example, consider the function

$$H(t, t_0) = \int_{t_0}^{t} e^{-\alpha(t-s)} \gamma(s) \, ds$$

and suppose that $\gamma(s) = e^{-\alpha s}$, which certainly satisfies (2.8). However,

$$\sup_{\sigma \geq 0} H(t + \sigma, t) e^{\alpha\sigma} = \sup_{\sigma \geq 0} e^{-\alpha t} \sigma e^{\alpha\sigma}$$

which, clearly, does not exist. This verifies our assertion.

4. Main Result

Here we shall study the preservation of almost sure stability behavior of ASI random set $\varphi = 0$ relative to (2.1) under constantly acting random perturbations. We consider the random functional perturbed system

$$y'(t) = f(t, y_t) + R(t, y_t), \qquad y_{t_0} = \varphi_0, \tag{4.1}$$

where $R \in C[R^+ \times C_\rho, R^n]$. Assume that (4.1) has unique solution throughout, given initial data.

THEOREM 4.1. *Assume that the hypotheses of Theorem 3.1 hold. Assume further that* $R(t, \varphi)$ *satisfies the estimate*

$$\| R(t, \varphi) \| \leq \lambda(t), \qquad t \in R^+, \quad \varphi \in C_\rho, \tag{4.2}$$

where $\lambda \in C[R^+, R^+]$ *and moreover*

$$\int_t^{t+1} [K(s, \tau(\omega)) \lambda(s) + \eta(s)] \, ds \to 0 \qquad \text{as} \quad t \to \infty \tag{4.3}$$

all $\omega \in \Omega$, *where* $K(t, \tau)$ *and* $\eta(t)$ *are as defined in the Theorem 3.1. Then*

(i) *the random set* $\varphi = 0$ *is* ASI *with respect to the system* (4.1);

(ii) *the* ASI *random set* $\varphi = 0$ *of the system* (4.1) *is almost sure equi-exponentially asymptotically stable.*

Proof: Define

$$H(t, t_0) = \int_{t_0}^{t} e^{-\alpha(t-s)} [K(s, \tau) \lambda(s) + \eta(s)] \tag{4.4}$$

where α, $K(t, \tau)$, $\lambda(t)$ and $\eta(t)$ are as defined in the theorem. In view of the relation (4.3), it is easy to observe that the function $H(t, t_0)$ has the desired properties. For details see [5]. Hence, we can choose $t_0 \geq T(\rho)$ such that

$$H(t, t_0) < \frac{\rho}{2}, \qquad \text{for} \quad t \geq t_0 \geq T(\rho). \tag{4.5}$$

Let $y_t(t_0, \varphi_0)$ be any solution of the perturbed system (4.1) such that

$$\| \varphi_0 \|_0 < \frac{\rho}{2K(t_0, \tau)} \qquad \text{and} \quad t_0 \geq T(\rho), \quad \text{w.p.l.} \tag{4.6}$$

Setting $\varphi = y_t(t_0, \varphi_0)$, we have

$$y_{t+h}(t_0, \varphi_0) = y_{t+h}(t, \varphi), \qquad h \geq 0,$$

because of the uniqueness of solutions. Suppose now that $x_{t+h}(t, \varphi)$, $h \geq 0$ is the solution of (2.1) through (t, φ). As long as $\| y_t(t_0, \varphi_0) \|_0 < \rho$ w.p.l. for $t \geq t_0 \geq T(\rho)$, we have, making use of the conclusions (*a*) and (*c*) of

Theorem 3.1, a Lyapunov functional $V(t, \varphi)$ such that

$$\begin{aligned} D^+ \underset{(2.2)}{V}(t, \varphi) &= \lim_{h \to 0^+} \sup \frac{1}{h} [V(t+h, y_{t+h}(t, \varphi)) - V(t+h, x_{t+h}(t, \varphi)) + \\ &+ V(t+h, x_{t+h}(t, \varphi)) - V(t, \varphi)] \leq \\ &\leq D^+ \underset{(2.1)}{V}(t, \varphi) + \lim_{h \to 0^+} \sup \frac{1}{h} [K(t+h, \tau) \| y_{t+h}(t, \varphi) - \\ &- x_{t+h}(t, \varphi) \|_0] \\ &\leq - \alpha V(t, \varphi) + \eta(t) + K(t, \tau) \| y'(t, \varphi)(t) - x'(t, \varphi)(t) \| \leq \\ &\leq - \alpha V(t, \varphi) + \eta(t) + K(t, \tau) \| R(t, \varphi) \| . \end{aligned}$$

Since $\varphi = y_t(t_0, \varphi_0)$, using the assumption (ii) and the property (b) of Theorem 3.1, we obtain

$$D^+ \underset{(2.2)}{V}(t, \varphi) \leq - \alpha V(t, \varphi) + \eta(t) + K(t, \tau) \lambda(t)$$

w.p.l. From this we get

$$(4.7) \qquad V(t, y_t(t, \varphi_0)) \leq V(t_0, \varphi_0) e^{-\alpha(t-t_0)} + \int_{t_0}^{t} [\lambda(s) + K(s, \tau) \lambda(s)] e^{-\alpha(t-s)} ds .$$

This together with the property (b) of Theorem 3.1 (4.5) and (4.6), we have

$$\| y_t(t_0, \varphi_0) \|_0 < \rho \qquad \text{for} \quad t > t_0 \geq T(\rho), \quad \text{w.p.l.}$$

Thus the estimate

$$\| y_t(t_0, \varphi_0) \|_0 \leq K(t_0, \tau) \| \varphi_0 \| e^{-\alpha(t-t_0)} + H(t, t_0), \qquad t > t_0$$

w.p.l. This implies that the random set $\varphi = 0$ is ASI with respect to (4.1) and it is almost sure equi-exponentially asymptotically stable. Hence the theorem is proved.

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