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Analisi matematica. — *Again on the strong solutions of the Navier-Stokes equations in three dimensional space* (*). Nota di GIOVANNI PROUSE, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Nella Nota II dal titolo *On the strong solutions of the Navier-Stokes equations in three dimensional space* la dimostrazione del Teorema 1 risulta incompleta in un punto. Si dà qui una dimostrazione leggermente modificata e semplificata di tale Teorema.

In Note II of the work "On the strong solutions of the Navier-Stokes equations in three dimensional space" the proof of Theorem 1 stated in Note I is incomplete at one point. In what follows we shall give a slightly modified and simplified proof of this Theorem, utilizing the same notations introduced in Note I. Setting

$$M = \max_{x \in \overline{\Omega}} |\vec{z}(x)| , \tag{1}$$

let us denote by $\vec{v}(x, t)$ the solution a.e. in Q of the Navier-Stokes inequalities relative to K_M (with $\vec{f} = 0$) satisfying the initial and boundary conditions (1.2), (1.7) of Note I; by the theorem proved in § 2 of Note I, such a solution exists and is unique.

We begin by proving the existence theorem; we shall show, for this, that $\vec{v}(x, t)$ *is also a solution a.e. in* Q *of the homogeneous Navier-Stokes equations satisfying* (1.2), (1,3).

By the assumptions made, $\vec{v}$ is such that

$$\vec{v}(t) \in L^2(0, T; N^1) \quad , \quad \vec{v}'(t) \in L^2(0, T; N^1) \quad , \quad \Delta\vec{v}(t) \in L^2(0, T; N^0) \tag{2}$$
$$\vec{v}(t) \in K_M \qquad \text{in } [0, T]$$

and, $\forall \vec{l}(t) \in L^2(0, T; N^0_{K_M})$,

$$\int_Q \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v} \times \text{grad})\,\vec{v}\right) \times (\vec{v} - \vec{l}) \, dQ \leq 0. \tag{3}$$

Denote now by Q′ the set ⊂ Q in which $|\vec{v}| = M$ and by Q″ the set Q—Q′; bearing in mind what was proved in § 1, Note I, the function $\vec{v}(x, t)$ is a solution of the Navier-Stokes equations a.e. in Q″, i.e.

$$\int_{Q''} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v} \times \text{grad})\,\vec{v}\right) \times \vec{h} \, dQ'' = 0 \tag{4}$$

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$\forall \vec{h}$ restriction to Q'' of a function $\in L^2(Q, T; N^0)$. Setting, as is obviously possible, $\vec{h} = \vec{v} - \vec{l}$ and subtracting (4) from (3), we obtain

$$(5) \qquad \int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu \Delta \vec{v} + (\vec{v} \times \text{grad}) \vec{v} \right) \times (\vec{v} - \vec{l}) \, dQ' \leq 0$$

$\forall \vec{l}$ restriction to Q' of a function $\in L^2(0, T; N^0_{KM})$.

Observe that, by the definitions given, $\frac{\partial \vec{v}}{\partial t}, \frac{\partial^2 \vec{v}}{\partial x_j^2} \in L^2(Q)$ and that, since $|\vec{v}|$ takes its maximum value on Q',

$$(6) \qquad -\int_{Q'} \Delta \vec{v} \times \vec{v} \, dQ' = -\int_{Q'} \sum_{j,k=1}^{3} \frac{\partial^2 v_j}{\partial x_k^2} v_j \, dQ' =$$

$$= -\int_{Q'} \sum_{j,k=1}^{3} \left[\frac{\partial}{\partial x_k} \left(\frac{\partial v_j}{\partial x_k} v_j \right) - \left(\frac{\partial v_j}{\partial x_k} \right)^2 \right] dQ' =$$

$$= \int_{Q'} \left[\sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k} \right)^2 - \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} \left(\frac{1}{2} |\vec{v}|^2 \right) \right] dQ' \geq \int_{Q'} \sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k} \right)^2 dQ' \geq 0,$$

$$\int_{Q'} \frac{\partial \vec{v}}{\partial t} \times \vec{v} \, dQ' = \frac{1}{2} \int_{Q'} \frac{\partial}{\partial t} |\vec{v}|^2 \, dQ' = 0,$$

$$\int_{Q'} (\vec{v} \times \text{grad}) \vec{v} \times \vec{v} \, dQ' = \int_{Q'} (\text{rot } \vec{v}) \wedge \vec{v} \times \vec{v} \, dQ' +$$

$$+ \frac{1}{2} \int_{Q'} \text{grad } |\vec{v}|^2 \times \vec{v} \, dQ' = 0.$$

Hence, substituting (6) into (5), we obtain

$$(7) \qquad \int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu \Delta \vec{v} + (\vec{v} \times \text{grad}) \vec{v} \right) \times \vec{l} \, dQ' \geq 0.$$

Since we can change $\vec{l}$ in $-\vec{l}$, from (7) it follows that

$$(8) \qquad \int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu \Delta \vec{v} + (\vec{v} \times \text{grad}) \vec{v} \right) \times \vec{l} \, dQ' = 0$$

$\forall \vec{l}$ restriction to Q' of a function $\in L^2(0, T; N^0_{KM})$. It is then obvious that (8) will hold $\forall \vec{l}$ restriction to Q' of a function $\in L^2(0, T; N^0)$.

Adding (4) and (8) we can conclude that $\vec{v}(x, t)$ is a solution a.e. in Q of the Navier-Stokes equations. The existence of a solution a.e. in Q satisfying conditions (1.2), (1.3) of Note I is therefore proved. The uniqueness of such a solution is, on the other hand, well known (see Prodi [6]; refer to the bibliography at the end of Note I).

Let us now prove the maximum principle expressed by Theorem 2 stated in Note I.

This follows directly from the proof of the preceding theorem. We have, in fact, shown that the solution in Q of the homogeneous Navier-Stokes equations belongs to K_M, i.e. that

$$|\vec{v}(x, t)| \leq M \qquad \text{a.e. in } Q,$$

where

$$M = \max_{x \in \overline{\Omega}} |\vec{z}(x)| .$$

The maximum principle is therefore proved.