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H. D. Pande, T. A. Khan

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Geometria differenziale. — *General decomposition of Berwald's curvature tensor fields in recurrent Finsler space.* Nota di H. D. PANDE e T. A. KHAN, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Estensione di alcuni risultati di Takano [3] al caso di spazî di Finsler ricorrenti in relazione ai tensori di curvatura secondo Berwald.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n with a positively homogeneous function $F(x, \dot{x})$ of degree one in $\dot{x}^i$ with a local coordinate system x^i. The metric tensor $g_{ij}(x, \dot{x}) = \frac{1}{2}\,\dot{\partial}_i\,\dot{\partial}_j\,F^2(x, \dot{x})$ is symmetric in its lower indices i and j and is positively homogeneous function of degree zero in $\dot{x}^k$. Let $T^i_j(x, \dot{x})$ be any tensor whose covariant derivative in the sense of Berwald [2] is given by

$$T^i_{j\,(k)} = \frac{\partial T^i_j}{\partial x^k} - \frac{\partial T^i_j}{\partial \dot{x}^l}\,\frac{\partial G^l}{\partial \dot{x}^k} + T^l_j\,G^i_{lk} - T^i_l\,G^l_{jk}. \tag{1.1}$$

The curvature tensor fields given by Berwald are

$$H^i_{jk} = \frac{\partial^2 G^i}{\partial x^k\,\partial \dot{x}^j} - \frac{\partial^2 G^i}{\partial x^j\,\partial \dot{x}^k} + G^i_{k\gamma}\,\frac{\partial G^\gamma}{\partial \dot{x}^j} - G^i_{\gamma j}\,\frac{\partial G^\gamma}{\partial \dot{x}^k}, \tag{1.2}$$

and

$$H^i_{jkh} = \frac{\partial G^i_{jk}}{\partial x^h} - \frac{\partial G^i_{jh}}{\partial x^k} + G^\gamma_{jk}\,G^i_{\gamma h} - G^\gamma_{jh}\,G^i_{\gamma k} + G^i_{\gamma jh}\,\frac{\partial G^\gamma}{\partial \dot{x}^k} - G^i_{\gamma jk}\,\frac{\partial G^\gamma}{\partial \dot{x}^h}, \tag{1.3}$$

where

$$G^i_{jkh} = \frac{\partial G^i_{kh}}{\partial \dot{x}^j}.$$

The curvature tensor fields H^i_j, H^i_{jk} and H^i_{jkh} hold the following identities: [2]

$$H^i_{j\,(k)} - H^i_{k\,(j)} + H^i_{jk\,(h)}\,\dot{x}^h = 0, \tag{1.4}$$

$$H^i_{jk\,(h)} + H^i_{kh\,(j)} + H^i_{hj\,(k)} = 0, \tag{1.5}$$

$$H^i_{jkh\,(l)} + H^i_{jhl\,(k)} + H^i_{jlk\,(h)} + H^\gamma_{kh}\,G^i_{\gamma jl} + H^\gamma_{lk}\,G^i_{\gamma jh} + H^\gamma_{hl}\,G^i_{\gamma jk} = 0, \tag{1.6}$$

$$H^i_{jkh} + H^i_{khj} + H^i_{hjk} = 0, \tag{1.7}$$

(*) Nella seduta del 15 dicembre 1973.

and

(1.8) $$\{ H^i_{jkh(l)} + H^i_{jlk(h)} + H^i_{jhl(k)} \} \dot{x}^l = H^\gamma_h G^i_{\gamma jk} - H^\gamma_k G^i_{\gamma jh} .$$

We have

(1.9) $$H_{ij} = H^h_{ijh} = \frac{\partial H_j}{\partial \dot{x}^i} ,$$

together with

$$H_i = H^h_{ih} .$$

The commutative formulae for the Berwald's curvature tensor fields are as follows:

(1.10) $$T_{(h)(k)} - T_{(k)(h)} = \frac{\partial T}{\partial \dot{x}^i} H^i_{hk} ,$$

(1.11) $$T^i_{j(h)(k)} - T^i_{j(k)(h)} = - \frac{\partial T^i_j}{\partial \dot{x}^\gamma} H^\gamma_{hk} + T^\gamma_j H^i_{\gamma hk} - T^i_\gamma H^\gamma_{jhk} ,$$

(1.12) $$\left(\frac{\partial T}{\partial \dot{x}^k} \right)_{(h)} - \left(\frac{\partial T_{(h)}}{\partial \dot{x}^k} \right) = 0 ,$$

and

(1.13) $$\left(\frac{\partial T^i_j}{\partial \dot{x}^k} \right)_{(h)} - \left(\frac{\partial T^i_{j(h)}}{\partial \dot{x}^k} \right) = T^i_\gamma G^\gamma_{jkh} - T^\gamma_j G^i_{\gamma kh} .$$

A Finsler space F_n is said to be Berwald's recurrent Finsler space if its Berwald's curvature tensor field satisfies the condition, [1]

(1.14) $$H^i_{jkh(l)} = V_l H^i_{jkh} ,$$

where V_l are the components of a recurrent vector field.

The following relations can be shown easily in view of (1.14)

(1.15) $$H^i_{kh(l)} = V_l H^i_{kh} ,$$

and

(1.16) $$H_{jk(l)} = V_l H_{jk} .$$

2. DECOMPOSITION OF BERWALD'S CURVATURE TENSOR FIELD H^i_{jkh}.

DEFINITION (2.1). We consider the decomposition of the Berwald's curvature tensor field H^i_{jkh} in the following form

(2.1) $$H^i_{jkh} = X^i_j \varphi_{kh}$$

where $\varphi_{kh}(x, \dot{x})$ and $X^i_j(x, \dot{x})$ are two tensor fields such that

(2.2) $$X^i_j V_i = p_j .$$

p_j is a non-zero vector field and is called a decompose vector field.

We prove the following theorems:

THEOREM 2.1. *In view of the decomposition* (2.1), *if the recurrence vector field* V_m *is independent of* $\dot{x}^i$, *the identities for* H^i_{jkh} *take the forms*

$$p_j \varphi_{kh} + p_k \varphi_{hj} + p_h \varphi_{jk} = 0, \tag{2.3}$$

and

$$V_j \varphi_{kh} + V_k \varphi_{hj} + V_h \varphi_{jk} = 0. \tag{2.4}$$

Proof. Because of relation (2.1), the equation (1.7) takes the form

$$X^i_j \varphi_{kh} + X^i_k \varphi_{hj} + X^i_h \varphi_{jk} = 0. \tag{2.5}$$

Multiplying V_i in (2.5) and using (2.2) we obtain (2.3).

Differentiating (1.15) with respect to $\dot{x}^i$ and applying the commutative formula (1.13) we have

$$H^{\gamma}_{kh} G^i_{\gamma jl} - H^i_{\gamma h} G^{\gamma}_{kjl} - H^i_{k\gamma} G^{\gamma}_{hjl} = 0. \tag{2.6}$$

Making a cyclic interchange in the indices k, h and l in (2.6) and adding the expressions thus obtained we have,

$$H^{\gamma}_{kh} G^i_{\gamma jl} + H^{\gamma}_{hl} G^i_{\gamma jk} + H^{\gamma}_{lk} G^i_{\gamma jh} = 0. \tag{2.7}$$

In view of the relations (1.6), (1.14), (2.1) and (2.7) we get

$$X^i_j (V_l \varphi_{kh} + V_k \varphi_{hl} + V_h \varphi_{lk}) = 0. \tag{2.8}$$

Multiplying V_i in (2.8) and using (2.2), we get the result (2.4).

COROLLARY 2.1. *If the recurrence vector field* V_m *is independent of* $\dot{x}^i$, *then the recurrent and decompose vector fields are identically equal in view of* (2.1).

Proof. It is obvious from the equations (2.3) and (2.4).

THEOREM (2.2). *Under the decomposition* (2.1), *if* V_m *is independent of the direction, the tensor fields* H^i_{jkh}, H^i_{jk} *and* φ_{kh} *satisfy the relation*

$$V_i H^i_{jkh} = V_h H_{jk} - V_k H_{jh} = p_j \varphi_{kh}. \tag{2.9}$$

Proof. Multiplying (2.1) by a recurrent vector field V_i and using the relation (2.2) we have,

$$V_i H^i_{jkh} = p_j \varphi_{kh}. \tag{2.10}$$

From the equation (2.3) and in view of the fact $\varphi_{jk} = -\varphi_{kj}$ we obtain

$$p_j \varphi_{kh} = p_k \varphi_{jh} - p_h \varphi_{jk}. \tag{2.11}$$

Contracting the equation (2.1) in the indices i and h we have,

$$H_{jk} = X^i_j \varphi_{ki}. \tag{2.12}$$

Transvecting (2.4) by X^j_l and using the relations (2.2) and (2.12) we get,

$$p_j \varphi_{kh} = V_h H_{jk} - V_k H_{jh}. \tag{2.13}$$

Combining the equations (2.10) and (2.13) we obtain the condition (2.9).

THEOREM. 2.3. *The recurrent vector field* V_i *and the tensor field* X^i_j *behave like a recurrent vector and a recurrent tensor field with respect to the decomposition* (2.1), *if* V_i *is independent of* $\dot{x}^i$ *and they are given by,*

$$V_{i(m)} = \lambda_m V_i, \tag{2.14}$$

and

$$X^i_{j(m)} = \mu_m X^i_j, \tag{2.15}$$

where λ_m *and* μ_m *are non-zero vector fields.*

Proof. Differentiating (2.9) covariantly with respect to x^m and using the relations (2.1) and (2.9) we obtain

$$V_{i(m)} X^i_j \varphi_{kh} = V_{h(m)} H_{jk} - V_{k(m)} H_{jh}. \tag{2.16}$$

Multiplying (2.16) by V_i and in view of (2.9) we get

$$V_{i(m)} (V_h H_{jk} - V_k H_{jh}) = (V_{h(m)} H_{jk} - V_{k(m)} H_{jh}) V_i. \tag{2.17}$$

Again multiplying V_a in (2.17) and using (2.17) we get

$$V_{i(m)} (V_h H_{jk} - V_k H_{jh}) V_a = V_{a(m)} (V_h H_{jk} - V_k H_{jh}) V_i. \tag{2.18}$$

That is

$$V_{i(m)} V_a = V_{a(m)} V_i, \tag{2.19}$$

provided

$$V_h H_{jk} - V_k H_{jh} \neq 0.$$

Which implies

$$V_{i(m)} = \lambda_m V_i,$$

where λ_m is a vector field.

Differentiating (2.2) covariantly with respect to x^m and using (2.14) we obtain

$$X^i_{j(m)} V_i = p_{j(m)} - \lambda_m p_j. \tag{2.20}$$

Again differentiating the equation (2.20) covariantly with respect to x^l and in view of (2.14) and (2.20) we get

$$X^i_{j(m)(l)} V_i + \lambda_l p_{j(m)} - \lambda_l \lambda_m p_j = p_{j(m)(l)} - \lambda_{m(l)} p_j - \lambda_m p_{j(l)}. \tag{2.21}$$

Subtracting from (2.21), the result thus obtained by interchanging the indices l and m in (2.21) we get,

$$(2.22)\qquad (X^i_{j\,(m)(l)} - X^i_{j\,(l)(m)})\, V_i = (p_{j\,(m)(l)} - p_{j\,(l)(m)}) - (\lambda_{m\,(l)} - \lambda_{l\,(m)})\, p_j\,.$$

Applying above the commutative formula (1.11) and using the fact $\frac{\partial}{\partial \dot{x}^\gamma}(X^i_j)\, V_i = \frac{\partial}{\partial \dot{x}^\gamma}(p_j)$ where V_i is independent of $\dot{x}^i$, we obtain

$$(2.23)\qquad p_\gamma\, H^\gamma_{jml} = (\lambda_{m\,(l)} - \lambda_{l\,(m)})\, p_j\,.$$

Multiplying V_γ in (2.23) and taking into account the relations (2.1) and (2.2) we get

$$(2.24)\qquad X^i_\gamma\, V_i\, \varphi_{ml} = (\lambda_{m\,(l)} - \lambda_{l\,(m)})\, V_\gamma\,.$$

Transvecting (2.24) by V_a and using the equation (2.24), we obtain

$$(2.25)\qquad X^i_\gamma\, V_i\, V_a\, \varphi_{ml} = X^i_a\, V_i\, V_\gamma\, \varphi_{ml}\,.$$

That is

$$(2.26)\qquad X^i_\gamma\, V_a = X^i_a\, V_\gamma\,.$$

Multiplying X^j_i in the equation obtained by differentiating (2.26) covariantly with respect to x^m and in view of equations (2.14) and (2.26) we have,

$$(2.27)\qquad X^j_a\, (X^i_{\gamma\,(m)} + \lambda_m\, X^i_\gamma) = (X^i_{a\,(m)} + \lambda_m\, X^i_a)\, X^j_\gamma\,.$$

Multiplying (2.27) by $V_i\, V_j$ and making use of (2.2) we get

$$(2.28)\qquad (X^j_a\, X^i_{\gamma\,(m)} - X^i_{a\,(m)}\, X^j_\gamma)\, V_i\, V_j = 0\,.$$

That is

$$(2.29)\qquad p_a\, X^i_{\gamma\,(m)} = X^i_{a\,(m)}\, p_\gamma\,,$$

which proves the result (2.15).

THEOREM 2.4. *Under the decomposition* (2.1) *the decomposed vector field* p_j *and the tensor field* φ_{kh} *behave respectively like a recurrent vector and recurrent tensor field as given by*

$$(2.30)\qquad p_{j\,(m)} = (\lambda_m + \mu_m)\, p_j\,,$$

and

$$(2.31)\qquad \varphi_{kh\,(m)} = (V_m - \mu_m)\, \varphi_{kh}\,.$$

Proof. Differentiating (2.2) covariantly with respect to x^m and with the help of the relations (2.14) and (2.15) we have the result (2.30).

Further differentiating (2.9) covariantly with respect to x^m and using the equations (1.15), (2.9), (2.14) and (2.30) we obtain (2.31).

COROLLARY 2.2. *If* $V_m = \mu_m = -\lambda_m$ *then the vector field* p_j *and the tensor field* φ_{kh} *are covariant constant.*

Proof. It is obvious from (2.30) and (2.31).

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