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Pseudo-projective curvature identities in Fins1er space

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Geometria differenziale. — *Pseudo-projective curvature identities in Finsler space.* Nota (*) di H. D. PANDE e A. KUMAR, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Estensione agli spazi di Finsler di identità relative ai campi vettoriali di deviazione e di curvatura pseudo-proiettiva.

1. INTRODUCTION

We consider an n-dimensional Finsler space F_n whose metric tensor defined by $g_{ij} = \frac{1}{2} \dot{\partial}^2_{ij} F^2(x, \dot{x})$, [1], is positively homogeneous of degree zero in $\dot{x}^i$, where $\dot{\partial}_i$ means $\partial/\partial\dot{x}^i$. The covariant derivative of a tensor field T^i_j is given by

$$T^i_{j,k} = \partial_k T^i_j - (\dot{\partial}_m T^i_j) G^m_k + T^m_j G^i_{mk} - T^i_m G^m_{jk}. \tag{1.1}$$

The deviation and projective deviation tensor fields $H^i_j(x, \dot{x})$ and $W^i_j(x, \dot{x})$ respectively are defined as follows:

$$H^i_k(x, \dot{x}) = 2\,\partial_k G^i - (\partial_h \dot{\partial}_k G^i)\,\dot{x}^h + 2\,G^i_{kl} G^l - (\dot{\partial}_l G^i)(\dot{\partial}_k G^l), \tag{1.2}$$

$$W^j_k(x, \dot{x}) = H^j_k - H\delta^j_k - (\dot{\partial}_i H^i_k - \dot{\partial}_k H)\,\dot{x}^j/(n+1) \tag{1.3}$$

where the function $G^i(x, \dot{x})$ and the tensor $H^i_k(x, \dot{x})$ is positively homogeneous of degree two in the $\dot{x}^i$, and

$$H^i_i \equiv (n-1)\,H.$$

2. PSEUDO-PROJECTIVE CURVATURE TENSORS

The pseudo-projective deviation tensor field $W^{*i}_j(x, \dot{x})$ [2], is given in the form

$$W^{*i}_j(x, \dot{x}) = aW^i_j + bH^i_j \tag{2.1}$$

where a and b are scalar functions depending on $(x, \dot{x})$ and homogeneous of degree zero in $\dot{x}^i$. The pseudo-projective curvature tensor fields are defined by

$$a)\quad W^{*i}_{hj}(x, \dot{x}) = \frac{2}{3}\dot{\partial}_{[h} W^{*i}_{j]}\ ^{(1)}\quad ,\quad b)\quad W^{*i}_{lhj}(x, \dot{x}) = \dot{\partial}_l W^{*i}_{hj}. \tag{2.2}$$

(*) Pervenuta all'Accademia il 24 luglio 1973.

(1) $2\,A_{[ij]} = A_{ij} - A_{ji}$ and $2\,A_{(ij)} = A_{ij} + A_{ji}$.

The following identities have been obtained: [3]

$$(2.3)\qquad a)\ W^{*i}_{i} = b\,(n-1)\,H \quad , \quad b)\ W^{*i}_{j}\,\dot{x}^{j} = 0$$

$$c)\ W^{*i}_{hj}\,\dot{x}^{h} = W^{*i}_{j} \quad , \quad d)\ W^{*i}_{lhj}\,\dot{x}^{l} = W^{*i}_{hj}.$$

With the help of equation (2.1), we obtain two more tensor fields $H^{i}_{jk}(x\,,\dot{x})$ and $H^{i}_{hjk}(x\,,\dot{x})$ defined similarly as in the equation (2.2). They are

$$(2.4)\qquad H^{i}_{hj}(x\,,\dot{x}) = \frac{2}{3}\left\{\dot{\partial}_{[h}\frac{1}{b}(W^{*i}_{j]} - aW^{i}_{j]}) - \frac{1}{b}(\dot{\partial}_{[h}\,aW^{i}_{j]})\right\} + $$
$$+ \frac{1}{b}\{W^{*i}_{hj} - aW^{i}_{hj}\}$$

and

$$(2.5)\qquad H^{i}_{lhj}(x\,,\dot{x}) = \frac{2}{3}\left\{\dot{\partial}^{2}_{l[h}\frac{1}{b}(W^{*i}_{j]} - aW^{i}_{j]}) + \dot{\partial}_{[h}\frac{1}{b}(\dot{\partial}_{\langle l\rangle}W^{*i}_{j]}\ ^{(2)} - \right.$$
$$- \dot{\partial}_{\langle l\rangle}a \# W^{i}_{j]} - a\dot{\partial}_{\langle l\rangle}W^{i}_{j]}) - \dot{\partial}_{l}\frac{1}{b}(\dot{\partial}_{[h}\,a \# W^{i}_{j]}) -$$
$$\left. - \frac{1}{b}(\dot{\partial}^{2}_{l[h}\,a \# W^{i}_{j]} + \dot{\partial}_{[h}\,a \# \dot{\partial}_{\langle l\rangle}W^{i}_{j]})\right\} +$$
$$+ \left\{\frac{1}{b}(W^{*i}_{lhj} - aW^{i}_{lhj} - (\dot{\partial}_{l}\,a)\,W^{i}_{hj}) + \dot{\partial}_{l}\frac{1}{b}(W^{*i}_{hj} - aW^{i}_{hj})\right\}.$$

Using equations (2.1), (2.2), (2.4) and (2.5), we get

$$(2.6)\qquad M_{l[h}\,W^{*i}_{j]} = S_{l[h}\,W^{i}_{j]},$$

where we put

$$(2.7)\qquad (A)\quad M_{lh}(x\,,\dot{x}) \overset{\text{def.}}{=} 2\left\{\dot{\partial}_{(l}\,b\,\dot{\partial}_{h)}\frac{1}{b}\right\}$$

and

$$(2.8)\qquad (B)\quad S_{lh}(x\,,\dot{x}) \overset{\text{def.}}{=} 2\left\{b\dot{\partial}_{(h}\frac{1}{b}\dot{\partial}_{l)}\,a + a\dot{\partial}_{(l}\,b\dot{\partial}_{h)}\frac{1}{b} + \frac{1}{b}\dot{\partial}_{(l}\,b\dot{\partial}_{h)}\,a\right\}.$$

With the help of equations (2.2) and (2.6), we obtain

$$(2.9)\qquad 3\,M_{lh}\,W^{*i}_{mj} = 3\,S_{lh}\,W^{i}_{mj} + 2\,\{\dot{\partial}_{m}\,S_{l[h}\,W^{i}_{j]} - \dot{\partial}_{j}\,S_{l[h}\,W^{i}_{m]} -$$
$$- \dot{\partial}_{m}\,M_{l[h}\,W^{*i}_{j]} + \dot{\partial}_{j}\,M_{l[h}\,W^{*i}_{m]} + S_{l[m}\,\dot{\partial}_{j]}\,W^{i}_{h} - M_{l[m}\,\dot{\partial}_{j]}\,W^{*i}_{h}\}\,,$$

and

$$(2.10)\qquad 3\,M_{lh}\,W^{*i}_{nmj} = 3\,\{S_{lh}\,W^{i}_{nmj} + (\dot{\partial}_{n}\,S_{lh})\,W^{i}_{mj} - (\dot{\partial}_{n}\,M_{lh})\,W^{*i}_{mj}\} +$$
$$+ 2\,\{(\dot{\partial}^{2}_{nm}\,S_{l[h)})\,W^{i}_{j]} - (\dot{\partial}^{2}_{nm}\,M_{l[h)})\,W^{*i}_{j]} - (\dot{\partial}^{2}_{nj}\,S_{l[h)})\,W^{i}_{m]} +$$
$$+ (\dot{\partial}^{2}_{nj}\,M_{l[h)})\,W^{*i}_{m]} + (\dot{\partial}_{m}\,S_{l[h)})\,\dot{\partial}_{\langle n\rangle}\,W^{i}_{j]} - (\dot{\partial}_{m}\,M_{l[h)})\,\dot{\partial}_{\langle n\rangle}\,W^{*i}_{j]} -$$
$$- (\dot{\partial}_{j}\,S_{l[h)})\,\dot{\partial}_{\langle n\rangle}\,W^{i}_{m]} + (\dot{\partial}_{j}\,M_{l[h)})\,\dot{\partial}_{\langle n\rangle}\,W^{*i}_{m]} + (\dot{\partial}_{n}\,S_{l[m)})\,\dot{\partial}_{j]}\,W^{i}_{h} +$$
$$+ S_{l[m}\,\dot{\partial}^{2}_{j]n}\,W^{i}_{h} - (\dot{\partial}_{n}\,M_{l[m)})\,\dot{\partial}_{j]}\,W^{*i}_{h} - M_{l[m}\,\dot{\partial}^{2}_{j]n}\,W^{*i}_{h}\}\,.$$

(2) The indices i and j in the notation $\langle\ \rangle$ are free from the symmetric and skew symmetric parts.

3. Pseudo-projective Veblen and Bianchi identities

We define the expression for the pseudo-projective Veblen identity in F_n as follows:

$$T^i_{lhnmjk} \overset{\text{def.}}{=} 3\, M_{lh} \{ W^{*i}_{nmj,k} + W^{*i}_{jnk,m} + W^{*i}_{kjm,n} + W^{*i}_{mkn,j} \} = 0. \tag{3.1}$$

Theorem (3.1). *In any Finsler space* F_n *the pseudo-projective Veblen identity is given by the equation* (3.2).

Proof. Differentiating covariantly equation (2.10) in the sense of Berwald and using (3.1) we obtain

$$T^i_{lhnmjk} = Q^i_{lhnmjk} + E^i_{lhnmjk} = 0, \tag{3.2}$$

where $Q^i_{lhnmjk}(x, \dot{x})$ is the sum of the first 28 terms containing W^{*i}_j, W^i_j, M_{lh}, S_{lh} and their derivatives and E^i_{lhnmjk} is defined the remaining last 96 terms in the right side of (3.1). The alternative form of (3.2) can be defined as follows:

$$T_{plhnmjk} = Q_{plhnmjk} + E_{plhnmjk} = 0 \tag{3.3}$$

as a consequence of lowering the indices.

Theorem (3.2). *The pseudo-projective Bianchi identities are*

$$\begin{aligned}
3\, M_{lh} (W^{*i}_{mj,k} + W^{*i}_{jk,m} + W^{*i}_{km,j}) &= 3 \{ S_{lh} (W^i_{mj,k} + \\
&+ W^i_{jk,m} + W^{*i}_{km,j}) + (S_{lh,k} W^i_{mj} + S_{lh,m} W^i_{jk} + \\
&+ S_{lh,j} W^i_{km}) - (M_{lh,k} W^{*i}_{mj} + M_{lh,m} W^{*i}_{jk} + M_{lh,j} W^{*i}_{km}) \} + \\
&+ 2 \{ (\dot\partial_m S_{l[h,\langle k\rangle}) W^i_{j]} + (\dot\partial_j S_{l[h,\langle m\rangle}) W^i_{k]} + (\dot\partial_k S_{l[h\langle j\rangle}) W^i_{m]} - \\
&- (\dot\partial_m M_{l[h,\langle k\rangle}) W^{*i}_{j]} - (\dot\partial_j M_{l[h,\langle m\rangle}) W^{*i}_{k]} - (\dot\partial_k M_{l[h,\langle j\rangle}) W^{*i}_{m]} + \\
&+ (\dot\partial_m S_{l[h} W^i_{j],k} + \dot\partial_j S_{l[h} W^i_{k],m} + \dot\partial_k S_{l[h} W^i_{m],j}) - \\
&- (\dot\partial_m M_{l[h} W^{*i}_{j],k} + \dot\partial_j M_{l[h} W^{*i}_{k],m} + \dot\partial_k M_{l[h} W^{*i}_{m],j}) - (\dot\partial_j S_{l[h,\langle k\rangle}) W^i_{m]} - \\
&- (\dot\partial_k S_{l[h,\langle m\rangle}) W^i_{j]} - (\dot\partial_m S_{l[h,\langle j\rangle}) W^i_{k]} + (\dot\partial_j M_{l[h,\langle k\rangle}) W^{*i}_{m]} + \\
&+ (\dot\partial_k M_{l[h,\langle m\rangle}) W^{*i}_{j]} + (\dot\partial_m M_{l[h,\langle j\rangle}) W^{*i}_{k]} - (\dot\partial_j S_{l[h} W^i_{m],k} + \\
&+ \dot\partial_k S_{l[h} W^i_{j],m} + \dot\partial_m S_{l[h} W^i_{k],j}) + (\dot\partial_j M_{l[h} W^{*i}_{m],k} + \\
&+ \dot\partial_k M_{l[h} W^{*i}_{j],m} + \dot\partial_m M_{l[h} W^{*i}_{k],j}) + S_{l[m,\langle k\rangle} \dot\partial_{j]} W^i_h + S_{l[j,\langle m\rangle} \dot\partial_{k]} W^i_h + \\
&+ S_{l[k,\langle j\rangle} \dot\partial_{m]} W^i_h - M_{l[m,\langle k\rangle} \dot\partial_{j]} W^{*i}_h - M_{l[j,\langle m\rangle} \dot\partial_{k]} W^{*i}_h - M_{l[k,\langle j\rangle} \dot\partial_{m]} W^{*i}_h + \\
&+ S_{l[m} \dot\partial_{j]} W^i_{h,k} + S_{l[j} \dot\partial_{k]} W^i_{h,m} + S_{l[k} \dot\partial_{m]} W^i_{h,j} - \\
&- M_{l[m} \dot\partial_{j]} W^{*i}_{h,k} - M_{l[j} \dot\partial_{k]} W^{*i}_{h,m} - M_{l[k} \dot\partial_{m]} W^{*i}_{h,j} \}
\end{aligned} \tag{3.4}$$

and

$$K^{i}_{lhnmjk} = X^{i}_{lhnmjk} + Y^{i}_{lhnmjk} = 0 \tag{3.5}$$

where

$$K^{i}_{lhnmjk} \overset{\text{def.}}{=} 3\, M_{lh} \{ W^{*i}_{nmj,k} + W^{*i}_{njk,m} + W^{*i}_{nkm,j} \} = 0 \tag{3.6}$$

and $X^{i}_{lhnmjk}(x, \dot{x})$ *is the sum of the first 21 terms containing* W^{*i}_{mj}, W^{i}_{mj}, W^{*i}_{nmj}, M_{lh} *and* S_{lh} *and their covariant derivatives,* $Y^{i}_{lhnmjk}(x, \dot{x})$ *is the remaining last 72 terms containing the terms of* S_{lh}, M_{lh} *and their derivatives.*

Proof. Differentiating equations (2.9) and (2.10) covariantly in the sense of Berwald, we easily obtain the identities (3.4) and (3.5).

From equations (3.2) and (3.5), we obtain

$$T^{i}_{lhnmjk} + T^{i}_{lhnjkm} + T^{i}_{lhnkmj} = K^{i}_{lhnmjk} + K^{i}_{lhjnkm} + K^{i}_{lhkjmn} + K^{i}_{lhmknj} + A^{i}_{lhnmjk}, \tag{3.7}$$

where $A^{i}_{lhnmjk}(x, \dot{x})$ contains S_{lh}, M_{lh}, W^{i}_{j}, W^{*i}_{j} and their covariant derivatives. Thus we have the following theorem:

THEOREM (3.3). *The necessary and sufficient condition that the pseudo Bianchi and Veblen identities can be expressed in terms of one another is that the tensor* $A^{i}_{lhnmjk}(x, \dot{x})$ *vanishes identically.*

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