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On antiinvolution geometry of a projective line over a field of characteristic > 2 and quadratic order

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Geometria. — *On antiinvolution geometry of a projective line over a field of characteristic* > 2 *and quadratic order* (*). Nota (**) di VINCENZO DICUONZO, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Oggetto di questa Nota è la costruzione gruppale di una classe di spazi metrici finiti mediante le antiinvoluzioni di certe rette proiettive finite e gli automorfismi interni involutori del gruppo generato dalle suddette antiinvoluzioni.

The purpose of this paper is to give a group construction of a class of finite metric spaces by the systems of antiinvolutions of certain finite projective lines and the involutorial inner automorphisms of the groups generated by the above systems of antiinvolutions.

Let r be a *projective line over a field* K *of characteristic* $p > 2$ *and quadratic order* $q = p^{2h}$; moreover let $\mathcal{T}$ be an *involutorial automorphism of* K. As it is known, K *contains a subfield* K$'$ *of order* $q' = p^h$, *consisting of the autoconjugate elements of* K. If we introduce a nonhomogeneous coordinate system on r and denote by $\bar{x}$ the conjugate of $x \in K$, the following equation $a_{11}\, x\bar{x}' + a_{12}\, x + a_{21}\, \bar{x}' + a_{22} = 0$, with $a_{ij} \in K$ and $a_{11}\, a_{22} - a_{12}\, a_{21} \neq 0$ represents an *antiprojectivity* on r; we call *antiinvolution* on r any involutorial antiprojectivity on r. There are $q'(q+1)$ antiinvolutions on r and each of them leaves $q' + 1$ points of r fixed, which form a so called *chain* of r (see bibliography [1]): we denote by Λ *the set of all antiinvolutions on* r.

We shall be concerned with some particularly important types of subsets of Λ later on: we introduce these now.

A *hyperbolic pencil of* Λ is the set of all antiinvolutions on r, which leave two distinct points of r fixed; a *parabolic pencil of* Λ is the set of all antiinvolutions, which leave a same point of r fixed; an *elliptic pencil of* Λ is the set of all antiinvolutions, which transform two distinct points of r onto one another. Because of the properties of the chains of r (see [1]), a pencil of Λ contains $q' + 1$, q', or $q' - 1$ antiinvolutions, according as it is hyperbolic, parabolic or elliptic; moreover the antiinvolutions, which commute with the ones of a pencil $\mathcal{F}$, form another pencil $\mathcal{F}^*$, which is hyperbolic, parabolic or elliptic, according as $\mathcal{F}$ is elliptic, parabolic or hyperbolic; furthermore there are $\frac{1}{2}\, q\,(q+1)$ hyperbolic pencils, as many as the elliptic ones, and $(q+1)(q'+1)$ parabolic pencils.

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At last we call *nonsingular bundle of* Λ the set of all antiinvolutions on r, which commute with a given element of Λ, and *singular bundle of* Λ the set of all antiinvolutions, which fix a given point of r; because of analogous properties of the chains, there are $q'(q+1)$ nonsingular and $q+1$ singular bundles of Λ and each of them contains q or $q+q'$ elements respectively.

Now we are going to construct a metric space Ω and to this aim we first assume *the antiinvolutions, the pencils and the bundles of* Λ as *planes, lines and points of the space* Ω, respectively.

A line of Ω is called *hyperbolic, parabolic* or *elliptic*, as the corresponding pencil of Λ; moreover *a point of* Ω is called *singular* or *nonsingular*, as the corresponding bundle of Λ.

If a bundle of Λ contains a pencil or an antiinvolution, we say " a point of Ω belongs to a line or a plane "; if an antiinvolution belongs to a pencil of Λ, we say " a plane of Ω contains a line or passes through a line ".

A line a and a plane β are said to be *incident* or *parallel*, according as the point $P = a \cap \beta$ is nonsingular or singular; likewise, two lines, passing through a point P, are called incident or parallel, according as P is nonsingular or singular.

Two planes of Ω are said to be *orthogonal*, if the corresponding antiinvolutions commute with one another.

By this definition and the one of nonsingular bundle of Λ, the planes, orthogonal to a same plane α, pass through a nonsingular point A, which is called the *pole* of α; conversely, the planes, through a nonsingular point A, are orthogonal to a plane α, which is called the *polar plane* of A.

Because of an above mentioned property of the pencils of Λ, the planes through a line a of Ω, are orthogonal to the ones through another line a', which is called the *polar line* of a; according as a is elliptic, parabolic or hyperbolic, a' is hyperbolic, parabolic or elliptic.

We now remark that *there can be no plane of* Ω *through three noncollinear points* A , B , C. In fact, this happens whether when A , B , C are poles of planes through a singular point, or when A is singular and B , C are poles of planes through A. Therefore we must introduce some new planes, which will be called isotropic.

As *isotropic plane relative to a singular point* P we assume *the set* π *consisting of* P *and the poles of the planes through* P.

By adding the isotropic planes we obtain a metric space, with the same incidence structure of projective three-space over K'.

Because of a property of orthogonal chains of r, *two orthogonal planes of* Ω *intersect in a line, which is either hyperbolic or elliptic.*

Now let α and β be two orthogonal planes passing through a nonparabolic line a, whose polar line is denoted by a': choosing two orthogonal planes γ and δ through a' we obtain four planes $\alpha , \beta , \gamma , \delta$ mutually orthogonal, which form an *autopolar tetrahedron* Δ. Such a tetrahedron is *of different type, according as* $\frac{1}{2}(q'+1)$ *is odd or even.*

In fact three edges of Δ, through a vertex, are hyperbolic or elliptic, according as $\frac{1}{2}(q'+1)$ is odd or even, and the other three ones are in a plane and all elliptic or hyperbolic respectively. This follows from a property of chains of r (see [1]).

We denote by $\bar{G}$ the group, whose generators are the elements of Λ, and by $\bar{\alpha}^*$ the involutorial inner automorphism of $\bar{G}$ relative to $\bar{\alpha} \in \Lambda$.

We now recall that the line r with its chains is an egglike inversive plane $\mathfrak{M}$. Since the product of three inversions of $\mathfrak{M}$, which are in a pencil $\bar{\mathfrak{F}}$, belongs to $\bar{\mathfrak{F}}$, likewise the product of three elements of Λ, which are in a pencil $\mathfrak{F}$, belongs to $\mathfrak{F}$.

It follows that Λ is invariant by inner automorphisms of $\bar{G}$ and $\bar{\alpha}^*$ transforms pencils and bundles of Λ onto pencils and bundles of the same type, preserves the commutativity for the elements of Λ and fixes whether $\bar{\alpha}$ or the antiinvolutions commuting with $\bar{\alpha}$. Denoting by α the plane relative to $\bar{\alpha}$, this means that $\bar{\alpha}^*$ transforms planes, lines and points of Ω onto planes, lines and points of the same type respectively, leaves all points and lines of α fixed and all planes and lines, orthogonal to α, invariant. Therefore we assume the involutorial inner automorphism $\bar{\alpha}^*$ of $\bar{G}$ as *plane reflection of* Ω.

Moreover we call *motion of* Ω *any inner automorphism of* $\bar{G}$, that is, any product of plane reflections of Ω.

Since each element of $\bar{G}$ may be represented as product of at most five generators, similarly *each motion of* Ω *may be represented as product of at most five plane reflections.*

We recall that $\bar{G}$ is isomorphic to the motion group of metric-projective three-space Σ of first type over K'; it follows that the motion group of Ω is isomorphic to the motion group of Σ. Therefore for other properties of motion group of Ω we refer to [2], n. 1.

For the planes of Ω the bundles of Λ enable us to give an independent group construction of them.

To this purpose we now consider a bundle $\mathfrak{R}$ of Λ and assume *the elements and the pencils of* $\mathfrak{R}$ as *lines and points of a plane* Π, respectively. According as $\mathfrak{R}$ is singular or nonsingular, we obtain two different types of planes, which will be denoted by Π_E and Π_H respectively.

We first suppose $\mathfrak{R}$ is *singular*. In this case a pencil of $\mathfrak{R}$ may be either parabolic or hyperbolic: correspondently a *point of* Π_E is called *singular* or *nonsingular*.

We now give the following definitions.

A point P of Π_E belongs to a line a, if the pencil corresponding to P contains the antiinvolution relative to a.

Two lines a and b of Π_E are *incident* or *parallel*, according as the antiinvolutions representing a and b determine a hyperbolic or parabolic pencil.

Two lines of Π_E are *orthogonal*, if the corresponding antiinvolutions commute with one another.

By a property of chains of r, the elements of $\mathfrak{R}$ commuting with an element α of $\mathfrak{R}$ form a parabolic pencil, whose elements commute with the ones of the parabolic pencil containing α; this means that *the lines of* Π_E, *orthogonal to a line a, are parallel and each of them is orthogonal to the lines parallel to a.* It follows that *there are rectangles in* Π_E.

We now denote by $\bar{G}_E$ *the group, whose generators are the elements of* $\mathfrak{R}$, and by $\bar{\alpha}^*$ *the involutorial inner automorphism of* $\bar{G}_E$ *relative to* $\bar{\alpha} \in \mathfrak{R}$.

If a is the line of Π_E corresponding to α, as we saw for plane reflections of Ω, $\bar{\alpha}^*$ leaves the line a pointwise fixed and the lines orthogonal to a invariant, and moreover transforms lines and points of Π_E into lines and points of the same type respectively. Therefore we call $\bar{\alpha}^*$ *line reflection of* Π_E; moreover we assume as *motion of* Π_E *any inner automorphism of* $\bar{G}_E$, that is, any product of line reflections.

We recall that the group $\bar{G}_E$ is isomorphic to the motion group $\mathcal{E}$ of Euclidean plane over K′ (see [1]): it follows that the motion group of Π_E is isomorphic to $\mathcal{E}$ and hence *any motion of* Π_E *may be represented as product of at most three line reflections* (see [2], n. 1).

At last we suppose *the bundle* $\mathfrak{R}$ *of* Λ *is nonsingular*, that is, the elements of $\mathfrak{R}$ commute with an antiinvolution λ of Λ. In this case $\mathfrak{R}$ contains $\frac{1}{2}q'(q'+1)$ elements having two fixed points of r in common with λ, and $\frac{1}{2}q'(q'-1)$ elements having no fixed point in common with λ: correspondently we call *hyperbolic* or *elliptic the lines of* Π_H.

Furthermore *a point of* Π_H is called *hyperbolic*, *elliptic* or *singular*, according as the corresponding pencil of $\mathfrak{R}$ is hyperbolic, elliptic or parabolic.

The definitions of incidence and orthogonality in Π_E *are valid for* Π_H *too.*

Because of definition of nonsingular bundle and a property of pencils of Λ, the elements of $\mathfrak{R}$ commuting with a given element α of $\mathfrak{R}$ form a nonparabolic pencil; conversely the elements of a nonparabolic pencil fo $\mathfrak{R}$ commute with an element of $\mathfrak{R}$. If follows that *the lines of* Π_H, *orthogonal to a line a, pass through a nonsingular point* A, *which is called pole of a*; conversely *the lines, through a nonsingular point* A, *are orthogonal to a line, which is called polar line of* A.

We now remark that *there can be no line of* Π_H *passing through two distinct points* A *and* B; in fact this happens when A and B are two nonsingular points, poles of two parallel lines a and b, or when A is a singular point and B is pole of a line through A. We can avoid this fact by introducing some so called isotropic lines.

If P is a *singular point* of Π_H, we assume as *isotropic line of* Π_H, *relative to* P, *the set consisting of* P *and the poles of the* (*hyperbolic*) *lines through* P.

Because of a property of the chains orthogonal to a chain of r (see [1]), *there are four types of triples of antiinvolutions in* $\mathfrak{R}$ *commuting two by two and in particular two of them, if* $\frac{1}{2}(q'+1)$ *is odd, while the other two, if* $\frac{1}{2}(q'+1)$ *is even*: precisely, if (α, β, γ) is such a triple, α, β, γ determine three pencils and the number of the hyperbolic ones is odd or even, according as $\frac{1}{2}(q'+1)$ is odd or even.

It follows that, *with respect to three-right-angled triangles, the planes* Π_H *may be divided into two classes, according as* $\frac{1}{2}(q'+1)$ *is odd or even*: precisely the number of the hyperbolic sides of such a triangle is odd or even, according as $\frac{1}{2}(q'+1)$ is odd or even.

As for Π_E, let G_H be *the group generated by* $\mathfrak{R}$ and denote by α^* the *involutorial inner automorphism of* G_H *relative to* $\alpha \in \mathfrak{R}$.

If a is the line of Π_H corresponding to α, as we saw for the plane reflections of Ω, α^* leaves the line a pointwise fixed, and the lines, orthogonal to a, invariant, and moreover transforms lines and points of Π_H onto lines and points of the same type respectively.

We call α^* *line reflection of* Π_H and assume as *motion of* Π_H *any inner automorphism of* G_H, that is, any product of line reflections.

The motions of Π_H *form a group* G_H^* *isomorphic to the motion group* $\bar{G}_H$ *of the hyperbolic plane over* K'. In fact G_H is isomorphic to $\bar{G}_H$ (see [1]).

It follows that *every motion of* Π_H *may be represented as product of two line reflections.*

For other properties of the motion group of Π_H we refer to [2], n. 1.

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