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An associativity criterion for Lie-Chernoff addition

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Semigruppi di operatori. — *An associativity criterion for Lie-Chernoff addition.* Nota di DAVID LOWELL LOVELADY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Se A, B e C sono generatori di semigruppi di operatori lineari fortemente continui non espansivi su uno spazio di Banach, si dice che è $C = A +_L B$ se, e solo se, $\exp[tC]x = \lim_{n\to\infty} (\exp[(t/n)A]\exp[(t/n)B])^n x$ per ogni (t, x). In questa Nota l'Autore dà un criterio perché l'operazione $+_L$ sia associativa.

Let **X** be a Banach space with norm | |, and let $R^+ = [0, \infty]$. Let **B**[**X**] be the algebra of continuous linear functions from **X** to **X**, and let I be the identity in **B**[**X**]. Let **G**[**X**] be the set to which A belongs if and only if A is the generator of a strongly continuous semigroup of members of **B**[**X**]. If (t, A) is in $R^+ \times$ **G**[**X**], let D(A) be the domain of A and let $\exp[tA]$ be the value at t of the semigroup generated by A.

In [2] (see also [1]), P. R. Chernoff has defined an additive process $+_L$ over **G**[**X**], the essential ingredient of which is that a generalized exponential identity hold: if A, B and C are in **G**[**X**] then we say $C = A +_L B$ if and only if

$$\exp[tC]\,x = \lim_{n\to\infty} (\exp[(t/n)A]\exp[(t/n)B])^n\, x$$

whenever (t, x) is in $R^+ \times$ **X**. Of course, it is not the case that $A +_L B$ always exists. Even worse, $+_L$ needs not be associative. It follows from [2, 5.3 Proposition] that even if X is a Hilbert space there are three members A, B and C of **G**[**X**] such that each of $A +_L (B +_L C)$ and $(A +_L B) +_L C$ exists, but their domains have trivial intersection.

In the present Note we shall give a criterion under which associativity does hold. If A and B are in **G**[**X**] then $A + B$ is the ordinary algebraic sum, defined on $D(A) \cap D(B)$, and $cl(A + B)$ is the closure of $A + B$.

THEOREM. *Suppose that each of* A, B *and* C *is in* **G**[**X**], *and that* $D(A) \cap D(B) \cap D(C)$ *is dense in* **X**. *Suppose also that each of* $cl(A+B)$, $cl(B+C)$, *and* $cl(A+B+C)$ *is in* **G**[**X**]. *Then each of* $A +_L (B +_L C)$ *and* $(A +_L B) +_L C$ *exists, and*

$$A +_L (B +_L C) = cl(A + B + C) = (A +_L B) +_L C. \tag{1}$$

(*) Nella seduta del 15 dicembre 1973.

Furthermore,

$$(2)\qquad \begin{aligned} &\exp\,[t(cl\,(A+B+C))]\,x \\ &= \lim_{n\to\infty}\,(\exp\,[(t/n)\,A]\,\exp\,[(t/n)\,B]\,\exp\,[(t/n)\,C])^n x \\ &= \lim_{n\to\infty}\,([I-(t/n)\,A]^{-1}\,[I-(t/n)\,B]^{-1}\,[I-(t/n)\,C]^{-1})^n x \end{aligned}$$

whenever $(t\,,x)$ *is in* $R^+ \times \mathbf{X}$.

Proof. Since each of $cl\,(A+B)$ and $cl\,(B+C)$ is in $\mathbf{G}[\mathbf{X}]$, it follows from [5, Lemma] that $A +_L B$ and $B +_L C$ exist and $cl\,(A+B) = A +_L B$, $cl\,(B+C) = B +_L C$. Now $cl\,(A+B)$, being in $\mathbf{G}[\mathbf{X}]$, is dissipative in the sense of G. Lumer and R. S. Phillips [3, Definition 1.2 and Theorem 3.1], so $cl\,(A+B)+C$ is dissipative and $cl\,(cl\,(A+B)+C)$ is dissipative [3, Lemma 3.4]. But $cl\,(A+B+C)$ is in $\mathbf{G}[\mathbf{X}]$ and $cl\,(cl\,(A+B)+C)$ is an extension of $cl\,(A+B+C)$, so $\mathbf{X} = ran\,(I - cl\,(A+B+C)) \subseteq$ $\subseteq ran\,(I - cl\,(cl\,(A+B)+C))$, and $cl\,(cl\,(A+B)+C)$ is in $\mathbf{G}[\mathbf{X}]$ [3, Theorem 3.1]. If one member of $\mathbf{G}[\mathbf{X}]$ extends another then the two are one, so $cl\,(cl\,(A+B)+C) = cl\,(A+B+C)$. Also, $cl\,(cl\,(A+B)+C) =$ $= cl\,((A +_L B)+C) = (A +_L B) +_L C$, so half of (1) is proved.

The other half of (1) follows analogously. The first part of (2) follows from [4, Theorem 5.3] via an argument analogous to that of [5, Lemma]. Let F be the strongly continuous function from R^+ to $\mathbf{B}\,[\mathbf{X}]$ given by

$$F(t) = [I - tA]^{-1}\,[I - tB]^{-1}\,[I - tC]^{-1}.$$

Routine computations now show that if x is in $D(A) \cap D(B) \cap D(C)$ then

$$\lim_{\delta\to 0+}\,(1/\delta)\,(F(\delta)\,x - x)$$

exists and equals $Ax + Bx + Cx$. Since each value of F is nonexpansive, it now follows from [4, Theorem 5.3] that

$$\exp\,[t\,(cl\,(A+B+C))]\;x = \lim_{n\to\infty} F(t/n)^n x$$

whenever $(t\,,x)$ is in $R^+ \times \mathbf{X}$. This completes the proof.

References

[1] P. R. Chernoff, *Semigroup product formulas and addition of unbounded operators*, « Bull. Amer. Math. Soc. », *76*, 395–398 (1970).

[2] P. R. Chernoff, *Product formulas, nonlinear semigroups, and addition of unbounded operators*, « Mem. Amer. Math. Soc. », 140 (1974).

[3] G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, « Pacific J. Math. », *11*, 679–698 (1961).

[4] H. F. Trotter, *Approximation of semi-groups of operators*, « Pacific J. Math. », *8*, 887–919 (1958).

[5] H. F. Trotter, *On the product of semi-groups of operators*, « Proc. Amer. Math. Soc. », *10*, 545–551 (1969).