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On the Fourier series with respect to S-integral

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Analisi funzionale. — *On the Fourier series with respect to S-integral.* Nota (*) di Yasusi Takemura, presentata dal Socio G. Sansone.

Riassunto. — L'Autore applicando la nozione di integrale S di Satō definisce le serie di Fourier di una funzione e ne studia alcune proprietà.

1. Introduction

The Fourier series had given mathematicians to reconsider strongly the notions of function and integral, and as its result the Riemann integral and the Lebesgue integral have been discovered.

Professor Satō introduced the notion of the S-integral in his paper [1].

Here, it is natural to ask the following questions.

Is it possible to define the Fourier series on the S-integral including the L-integral?

Is it possible for us to find analytical meanings of this formal series? In this paper, we concern with these problems.

2. Notation

We shall use some notations from [1].

Let $f(x) \in S(a, b)$, then we write

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx \quad , \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

A function $f(x)$ belongs to $S_0(a, b)$, if and only if the function belongs to $S(a, b)$, and

$$\lim_{\lambda \to \infty} \lambda\mu(e_\lambda) = 0$$

where $e_\lambda = \{x : |f(x)| > \lambda, x \in [a, b]\}$.

A function $f(x)$ belongs to $S_1(a, b)$ if and only if the function is S-integrable in (α, β) whenever $a \leqq \alpha < \beta \leqq b$.

Let $f(x) \in S_1(a, b)$, then the function

$$F(x) = \int_a^x f(x)\,dx \qquad (F(a) = {}_d 0)$$

is called the indefinite integral of $f(x)$.

(*) Pervenuta all'Accademia il 25 settembre 1973.

Let $f(x) \in S_1(a, b)$. A function $f(x)$ belongs to $S_2(a, b)$ if and only if the indefinite integral $F(x)$ has almost everywhere a finite differential coefficient equal to $f(x)$.

This is written in the form of

$$F'(x) \doteq f(x) \qquad \text{for} \quad x \in [a, b].$$

We write

$$S_3(a, b) = \{f(x) : F(x) \in C^0([a, b]), f(x) \in S_2(a, b)\}.$$

Suppose that a function $g(x)$ is absolutely continuous in $[a, b]$.
Then we define

$$S_4(a, b) = \{f(x) : f(x) g(x) \in S_3(a, b), f(x) \in S_3(a, b)\}.$$

Let $f(x) \in S_4(a, b)$.
Suppose that a function $g(x)$ is absolutely continuous in $[a, b]$.
A function $f(x)$ belongs to $S_5(a, b)$ if and only if we have

$$F(x) g(x) = \int_a^x \frac{d}{dx}(F(x) g(x)) \, dx,$$

where $F(x)$ is the indefinite integral of $f(x)$.

Further we shall use the following notation:

$$S_{01}(a, b) = S_0(a, b) \cap S_1(a, b),$$

$$S_{02}(a, b) = S_{01}(a, b) \cap S_2(a, b),$$

$$S_{03}(a, b) = S_{02}(a, b) \cap S_3(a, b),$$

$$S_{04}(a, b) = S_{03}(a, b) \cap S_4(a, b),$$

$$S_{05}(a, b) = S_{04}(a, b) \cap S_5(a, b).$$

3. Let $f(x) \in S_{04}(a, b)$. We can define

(1) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 0, 1, 2, \cdots).$$

The numbers a_n, b_n are called the Fourier coefficient of $f(x)$. Then we may consider the formal series:

(2) $$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

This formal series is called the Fourier series of $f(x)$. Without stated the contrary, each function has the period 2π and it is finite.

Let $0 \leqq r < 1$.

We shall use the notation:

$$P_r(t) = \frac{1-r^2}{2(1-2r\cos t + r^2)} .$$

$P_r(t)$ is known as Poisson's kernel.

$P_r(t)$ is an even function with period 2π for $-\infty < t < +\infty$, moreover it has the following property:

$$P_r(t) > 0 ,$$

$$P'_r(t) \begin{cases} \geqq 0 & -\pi \leqq t \leqq 0 \\ \leqq 0 & 0 \leqq t \leqq \pi . \end{cases}$$

Let $0 \leqq r < 1$.

We have

$$\frac{r^n \cos nu - r^{n+1}\cos(n-1)u}{1-2r\cos u + r^2} = r^n \cos nu + \frac{r^{n+1}\cos(n+1)u - r^{n+2}\cos nu}{1-2r\cos u+r^2}$$

$$(n = 1, 2, \cdots),$$

hence we obtain

$$\frac{1-r^2}{1-2r\cos u+r^2} = 1 + 2\sum_{k=1}^{n} r^k \cos ku + \frac{2(r^{n+1}\cos(n+1)u - r^{n+2}\cos nu)}{1-2r\cos u + r^2} .$$

Let $u = t - x$, then we have

$$P_r(t-x) = \frac{1}{2} + \sum_{k=1}^{n} (\cos kt \cos kx + \sin kt \sin kx) r^k +$$

$$+ \frac{r^{n+1}\cos(n+1)(t-x) - r^{n+2}\cos n(t-x)}{1-2r\cos(t-x)+r^2} .$$

By

$$\left| \frac{r^{n+1}\cos(n+1)(t-x) - r^{n+2}\cos n(t-x)}{1-2r\cos(t-x)+r^2} \right| \leqq \frac{r^{n+1}(1+r)}{(1-r)^2} \qquad (n = 1, 2, \cdots),$$

we have

$$\lim_{n\to\infty} \frac{r^{n+1}\cos(n+1)(t-x) - r^{n+2}\cos n(t-x)}{1-2r\cos(t-x)+r^2} = 0.$$

The left side is uniformly convergent in $-\infty < t < +\infty$.

$P_r(t-x)$ may be expanded as the following

$$P_r(t-x) = \frac{1}{2} + \sum_{n=1}^{\infty} (\cos nt \cos nx + \sin nt \sin nx) r^n. \tag{3}$$

The series of the right side is uniformly convergent in $-\infty < t < +\infty$, $-\infty < x < +\infty$, $0 \leqq r < 1$.

THEOREM 1. *Let* $f(x) \in S_{04}(-\pi, \pi)$.

Let (2) *be the Fourier series of* $f(x)$. *If the sequences* $\{a_n\}$, $\{b_n\}$ *are bounded, then we have the following identity*:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t) P_r(t-x)\,dt = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n \quad \text{for } 0 \leqq r < 1. \tag{4}$$

Proof. Let $x \in R^{\#}$, $0 \leqq r < 1$.
Then we have the following relation:

$$| a_0 |/2 + \sum_{n=1}^{\infty} | a_n \cos nx + b_n \sin nx | r^n$$

$$\leqq | a_0 |/2 + \sum_{n=1}^{\infty} (| a_n | + | b_n |) r^n$$

$$\leqq | a_0 |/2 + 2\, Mr/1 - r$$

where M is a constant such that

$$| a_n | , | b_n | \leqq M \qquad (n = 1 , 2 , \cdots).$$

The series of the left side is uniformly convergent in $-\infty < x < +\infty$.
By (3) and the Theorem 2.1 in [3], we have

$$\left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, P_r(t-x)\, dt - \left(a_0/2 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)\, r^k \right) \right|$$

$$\leqq \sum_{k=n+1}^{\infty} \left| \left(\left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\, dt \right) \cos kx + \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\, dt \right) \sin kx \right) r^k \right|$$

$$\leqq \sum_{k=n+1}^{\infty} (| a_k | + | b_k |) r^k$$

$$\leqq 2\, Mr^{n+1}/(1-r) .$$

Let $n \to \infty$, then we obtain (4).

The following corollary can be obtained from the well known Riemann-Lebesgue's theorem.

COROLLARY (Fatou). *Let* $f(x) \in L(-\pi , \pi)$.
Let (2) *be the Fourier series of* $f(x)$. *Then we have* (4).

THEOREM 2. *Let* $f(x) \in S_{05}(a , b)$.
Let $g(x)$ *be monotone increasing and absolutely continuous in* $[a , b]$.
Then there exists a ξ *such that*

$$\int_a^b f(x)\, g(x)\, dx = g(a) \int_a^{\xi} f(x)\, dx + g(b) \int_{\xi}^{b} f(x)\, dx \qquad \textit{where} \quad a \leqq \xi \leqq b .$$

Proof. By $g'(x) \stackrel{\cdot}{\geqq} 0$ and the Theorem 3.12 in [2], we have

$$\int_a^b f(x)\, g(x)\, dx = F(b)\, g(b) - \int_a^b F(x)\, g'(x)\, dx, \qquad \text{where} \quad F(x) = \int_a^x f(x)\, dx.$$

Put

$$\int_a^b F(x) g'(x) \, dx = \mu \int_a^b g'(x) \, dx, \qquad \text{where} \quad m \leqq \mu \leqq M \,,$$

$m = \inf_{x \in [a,b]} F(x)$, $M = \sup_{x \in [a,b]} F(x)$. Since $F(x)$ is continuous in $[a, b]$, it follows that $F(\xi) = \mu$ for some ξ satisfying $a \leqq \xi \leqq b$.

Hence we have

$$\int_a^b F(x) g'(x) \, dx = F(\xi) (g(b) - g(a)).$$

Therefore we obtain the theorem.

THEOREM 3. *Let* $f(x) \in S_{05}(-\pi, \pi)$.

Let (2) *be the Fourier series of* $f(x)$.

If the sequences $\{a_n\}$, $\{b_n\}$ *are bounded and*

$$\int_0^t (\varphi(x, t) - S) \, dt = o(t) \,, \tag{5}$$

then the Fourier series (2) *of* $f(x)$ *is summable* (A) *to sum* S *for every value of* x, *where*

$$\varphi(x, t) = (f(x+t) + f(x-t))/2 \,.$$

Proof. It is easy to see that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} P_r(x) \, dx = 1 \quad , \quad \int_0^{\pi} t \,|\, P_r'(t) \,|\, dt \leqq \pi/2.$$

By the periodicity of $f(x)$ and the Theorem 4.3 in [3], we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, P_r(t - x) \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t + x) \, P_r(t) \, dt$$

and since $P_r(t)$ is an even function, we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \, P_r(t) \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t) \, P_r(t) \, dt.$$

Let $r \in [0, 1]$.

$P_r(t)$ is absolutely continuous in $[0, \pi]$.

By the assumption, we have

$$(\varphi(x, t) - S) \in S_{05}(-\pi, \pi) \qquad \text{for } t \,, \tag{6}$$

$$(\varphi(x, t) - S) \, P_r(t) \in S_{05}(-\pi, \pi) \qquad \text{for } t \,. \tag{7}$$

We define

$$\Phi(t) = \int_0^t (\varphi(x, t) - S)\, dt \qquad (-\pi \leqq t \leqq \pi).$$

Then $\Phi(t)$ is continuous in $[-\pi, \pi]$.

By the Theorem 1,

$$(8) \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} (\varphi(x, t) - S)\, P_r(t)\, dt = \frac{2}{\pi}\int_0^{\pi} (\varphi(x, t) - S)\, P_r(t)\, dt =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, r^n - S.$$

By (5), if we give an arbitrary $\varepsilon > 0$, then there is a $\delta > 0$ such that

$$(9) \qquad |\Phi(t)| \leqq \varepsilon t \qquad (0 \leqq t \leqq \delta < \pi).$$

For such δ, by (7), we may consider

$$I_1 = \int_0^{\delta} (\varphi(x, t) - S)\, P_r(t)\, dt \quad , \quad I_2 = \int_{\delta}^{\pi} (\varphi(x, t) - S)\, P_r(t)\, dt.$$

By the Theorem 3.12 in [2], we have

$$I_1 = \Phi(\delta)\, P_r(\delta) - \int_0^{\delta} \Phi(t)\, P_r'(t)\, dt.$$

It follows from (9) that

$$|I_1| \leqq \frac{1 - r^2}{(1 - r)^2 + 4r\sin^2\dfrac{\delta}{2}}\, \Phi(\delta) + \varepsilon \int_0^{\delta} t P_r'(t)\, dt \leqq$$

$$\leqq \left\{ \frac{(1 - r^2)\,\delta}{(1 - r)^2 + 4r\sin^2\dfrac{\delta}{2}} + \frac{\pi}{2} \right\} \varepsilon.$$

$P_r(x)$ is monotone increasing and absolutely continuous.

Hence we obtain

$$I_2 = P_r(\delta) \int_{\delta}^{\xi} (\varphi(x, t) - S)\, dt + P_r(\pi) \int_{\xi}^{\pi} (\varphi(x, t) - S)\, dt \qquad (\delta \leqq \xi \leqq \pi).$$

Since $\Phi(t)$ is continuous, there exists a constant M such that

$$|\Phi(t)| \leqq M \qquad \text{for} \quad t \in [0, \pi].$$

We have

$$|\mathrm{I}_2| \leqq 2\,(\mathrm{P}_r(\delta) + \mathrm{P}_r(\pi))\,\mathrm{M}\,,$$

$$\begin{aligned}\lim_{r\to1-0}\left|\int_{-\pi}^{\pi}(\varphi(x,t)-\mathrm{S})\,\mathrm{P}_r(t)\,\mathrm{d}t\right| &\leqq \lim_{r\to1-0}|\mathrm{I}_1| + \lim_{r\to1-0}|\mathrm{I}_2| \\ &\leqq \pi\varepsilon/2\,.\end{aligned}$$

Since ε is arbitrary, it follows that

$$\lim_{r\to1-0}\int_{-\pi}^{\pi}(\varphi(x,t)-\mathrm{S})\,\mathrm{P}_r(t)\,\mathrm{d}t = 0\,.$$

By (8), (2) is summable (A) to sum S.

The following corollary follows from the Theorem 3.8 in [2].

COROLLARY. *Let* $f(x) \in \mathrm{S}_{05}(-\pi, \pi)$.

Let (2) *be the Fourier series of* $f(x)$.

If the sequences $\{a_n\}$, $\{b_n\}$ *are bounded, the Fourier series* (2) *of* $f(x)$ *is summable* (A) *to sum* $f(x)$ *almost everywhere.*

REFERENCES

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