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Partial differential equations in an unbounded medium

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Equazioni a derivate parziali. — *Partial differential equations in an unbounded medium.* Nota [(*)] di M. MOHAMED EL BORAI, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Viene qui dato un teorema del tipo di Liouville per alcune generali equazioni a derivate parziali d'ordine superiore.

§ 1. INTRODUCTION

The harmonic function which tends to zero at infinity is identically zero. This is the known Liouville's fact [5].

In this note we are introducing a Liouville's type theorem for some general partial differential equations of higher orders.

The existence of solutions for these equations is proved and its general forms are obtained.

§. 2 A THEOREM OF LIOUVILLE'S TYPE

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a n–tuple of non-negative integers and let:

$$D^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad , \quad \xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n} \, , \, |\alpha| = \sum_{j=1}^{n} \alpha_j$$

where $x = (x_1, \cdots, x_n)$, $\xi = (\xi_1, \cdots, \xi_n)$ are vectors in the n–Euclidean space E_n.

By $C_m(E_n)$ we shall denote the class of complex functions on E_n with continuous derivatives up to order m.

Let $Q(x, D) u(x) = \sum_{|\alpha| = 2m} b_\alpha(x) D^\alpha u$ be a linear partial differential operator of homogeneous order $2m$ with complex bounded measurable coefficients $b_\alpha(x)$ on E_n.

Suppose that $P(D) u = \sum_{|\alpha| = 2m} a_\alpha D^\alpha u$ is an elliptic operator of homogeneous order $2m$ with complex constant coefficients a_α.

If $L_p(E_n)$ is the space of summable *p-th* power function of E_n, denote by $L_p^{2m}(E_n)$ the subset of $L_p(E_n)$ of all functions which have all derivatives of order less than or equal to $2m$ in the space $L_p(E_n)$.

We shall use the usual notation

$$\| f(x) \|_{L_p(E_n)} = \left[\int_{E_n} | f(x) |^p \, dx \right]^{1/p} .$$

(*) Pervenuta all'Accademia il 10 ottobre 1973.

Let $R_{j\varepsilon}$ be the operator, defined by:

$$R_{j\varepsilon} f(x) = -i\pi^{\frac{-(n+1)}{2}} \Gamma\left(\frac{n+1}{2}\right) \int\limits_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy ,$$

where $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$, $f(x) \in L_p(E_n)$.

Assume that R_j is the Calderon-Zygmond operator defined by,

$$\lim_{\varepsilon \to 0} \| (R_j - R_{j\varepsilon}) f(x) \|_{L_p(E_n)} = 0.$$

Compare Calderon-Zygmond [2].

If $v(x) \in L_p^{2m}(E_n)$, $p \geq 2$, then according to the theory of singular integral operators we deduce that

(1)
$$P(D)\, v(x) = \sum_{|\alpha|=2m} a_\alpha R^\alpha \nabla^{2m} v(x) \cdots$$

(2)
$$Q(x, D)\, v = \sum_{|\alpha|=2m} b_\alpha(x) R^\alpha \nabla^{2m} v(x),$$

where

$$\nabla^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad , \quad R^\alpha = R_1^{\alpha_1} R_2^{\alpha_2} \cdots R_n^{\alpha_n} .$$

We shall prove now an elementary Lemma, which will be useful in the sequence.

LEMMA. *There exists a positive integer p such that*

$$\lim_{\varepsilon \to 0} \int\limits_{E_n} e^{-\varepsilon |x|^2} \varepsilon^{kp} |x|^{mp} dx = 0, \text{ if } k > \frac{m}{2} .$$

Proof. Let Ω_n be the volume of a unit sphere in the n–dimensional space E_n.

If

$$r^2 = \sum_{j=1}^{n} x_j^2 = |x|^2 ,$$

then it is clear that:

$$\int\limits_{E_n} e^{-\varepsilon|x|^2} \varepsilon^{kp} |x|^{mp} dx = \Omega_n \int_0^\infty e^{-\varepsilon r^2} \varepsilon^{kp} r^{mp+n-1} dr =$$

$$= \varepsilon^{p\left(k - \frac{m}{2}\right)} \varepsilon^{\frac{-(n-2)}{2}} \int_0^\infty e^{-r^2} r^{mp+n-1} dr ,$$

and the result follows.

THEOREM 1. *Let* $u_\lambda(x) \in C_{2m}(E_n)$ *be a solution of the following equation*

$$P(D)\, u_\lambda(x) + \lambda Q(x, D)\, u_\lambda(x) = 0 \tag{3}$$

if $\sum_{|\alpha|=j} |D^\alpha u_\lambda(x)| \leq c(1+|x|)^{2m-j-1}$, *then for sufficiently small* $|\lambda|$ *the solution* $u_\lambda(x)$ *is a polynomial of degree* $\leq 2m-1$, $j = 0, 1, \cdots, 2m-1$.

Proof. Let $V_\varepsilon(x) = e^{-\varepsilon|x|^2} u_\lambda(x)$, where ε is a positive number.
It is easy to see that:

$$[P(D) + \lambda Q(x, D)]\, V_\varepsilon(x) = e^{-\varepsilon|x|^2} [P(D)\, u_\lambda(x) + \lambda Q(x, D)\, u_\lambda(x)] + \\ + e^{-\varepsilon|x|^2} L(\varepsilon, x, D)\, u_\lambda(x). \tag{4}$$

Where $L(\varepsilon, x, D)$ is a differential operator of order $2m-1$. If $\xi \neq 0$ is any real vector then $L(\xi, x, \xi)$ represents a polynomial of ξ of degree $2m$. Using (3) and (4) we get:

$$[P(D) + \lambda Q(x, D)]\, V_\varepsilon(x) = e^{-\varepsilon|x|^2} L(\varepsilon, x, D)\, u_\lambda(x). \tag{5}$$

If $p \geq 2$, then it is clear that $v_\varepsilon(x) \in L_p^{2m}(E_n)$. According to the theory of singular integral operator [1], we can deduce from (5) that;

$$(H + \lambda G)\, \nabla^{2m} V_\varepsilon(x) = e^{-\varepsilon|x|^2} L(\varepsilon, x, D)\, u_\lambda(x) \cdots \tag{6}$$

where:

$$H = \sum_{|\alpha|=2m} a_\alpha R^\alpha \quad , \quad G = \sum_{|\alpha|=2m} b_\alpha(x) R^\alpha \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n .$$

Since $\sum_{|\alpha|=2m} a_\alpha \xi^\alpha |\xi|^{-2m} \neq 0$, for any real vector $\xi \neq 0$, then there exists a bounded inverse operator H^{-1} in the space $L_n(E_p)$. Consequently equation (6) can be written in the form

$$(I + \lambda H^{-1} G)\, \nabla^{2m} V_\varepsilon(x) = H^{-1} [e^{-\varepsilon|x|^2} L(\varepsilon, x, D)\, u_\lambda(x)].$$

If λ is a sufficiently small number so that $|\lambda|\, \| H^{-1} G \|_{L_p(E_n)} < 1$, then there exists a bounded inverse operator $(I + H^{-1} G)^{-1}$ in the space $L_p(E_n)$.
Thus we have:

$$\nabla^{2m} V_\varepsilon(x) = (I + \lambda H^{-1} G)^{-1} H^{-1} [e^{-\varepsilon|x|^2} \cdot L(\varepsilon, x, D)\, u_\lambda(x)]. \tag{7}$$

Since the singular integral operators H^{-1} and G are bounded in the space $L_p(E_n)$, then there exists a positive constant K such that

$$\| (I + \lambda H^{-1} G)^{-1} H^{-1} [e^{-\varepsilon|x|^2} L(\varepsilon, x, D)\, u_\lambda(x)] \|_{L_p(E_n)} \leq \\ \leq K \| e^{-\varepsilon|x|^2} L(\varepsilon, x, D)\, u_\lambda(x) \|_{L_p(E_n)}. \tag{8}$$

Using (7) and (8), we obtain

$$\| \nabla^{2m} V_\varepsilon(x) \|_{L_p(E_n)} \leq K \| e^{-\varepsilon|x|^2} L(\varepsilon, x, D) u_\lambda(x) \|_{L_p(E_n)} . \tag{9}$$

Let S_A denote the volume enclosed by the the sphere $\sum_{j=1}^{n} x_j^2 = A^2$. Now inequality (9) gives

$$\| \nabla^{2m} V_\varepsilon(x) \|_{L_p(S_A)} \leq K \| e^{-\varepsilon|x|^2} L(\varepsilon, x, D) u_\lambda(x) \|_{L_p(E_n)} . \tag{10}$$

If $|x| \geq 1$, it can be easily proved that

$$| L(\varepsilon, x, \xi) | \leq \sum_{j=0}^{2m-1} K_j \varepsilon^{2m-j} r^{2m-j}, \qquad |\xi| \leq 1$$

where $K_j, j = 0, 1, 2, \cdots, 2m-1$ are some positive constants.

Since $\sum_{|\alpha|=j} | D^\alpha u_\lambda(x) | \leq c(1+|x|)^{2m-j-1}$, then according to the previous lemma we can find an integer, $p \geq 2$, such that

$$\lim_{\varepsilon \to 0} \| \nabla^{2m} V_\varepsilon(x) \|_{L_p(S_A)} \leq \lim_{\varepsilon \to 0} K \| e^{-\varepsilon|x|^2} L(\varepsilon, x, D) u_\lambda \|_{L_p(E_n)} = 0 .$$

Consequently:

$$\lim_{\varepsilon \to 0} \| \nabla^{2m} V_\varepsilon(x) \|_{L_p(S_A)} = \| \nabla^{2m} u_\lambda(x) \|_{L_p(S_A)} = 0 .$$

This means that $\nabla^{2m} u_\lambda(x) = 0$ for any real vector $x \in S_A$. Since there is no restriction on the positive number A, we can deduce that $\nabla^{2m} u_\lambda(x) = 0$, for any real vector x. According to the conditions of the theorem we deduce that $u_\lambda(x)$ is a polynomial of degree less than or equal to $2m-1$.

Remark. If $\sum_{|\alpha|=j} \| D^\alpha u_\lambda(x) | \leq c(1+|x|)^{l-j}$, $l \leq 2m-1$, then $u_\lambda(x)$ is a polynomial of degree $\leq [l]$, where $[l]$ is the least positive integer in l.

The proof of this result runs parallel to the proof of the theorem.

COROLLARY 1. *If* $\sum_{|\alpha| \leq 2m} | D^\alpha u_\lambda(x) | < c$, *then for sufficiently small* $|\lambda|$ *the solution* $u_\lambda(x) =$ *constant.*

COROLLARY 2. *If* $\lim_{r \to \infty} \sum_{|\alpha| \leq 2m} | D^\alpha u_\lambda(x) | = 0$ *then for sufficiently small* $|\lambda|$ *the solution* $u_\lambda(x)$ *is identically equal to zero. (The number* c *represents a positive constant).*

§ 3. THE SOLUTION IN THE SPACE $L_2(E_n)$

Let $W_2^{2m}(E_n)$ be the subset of $L_2(E_n)$ of all functions which have all generalized derivatives (in the sense of Sobolev) of order $\leq m$ in $L_2(E_n)$ [3], [4].

Consider the partial differential equation:

$$P(D) u_\lambda(x) + \lambda Q(x, D) u_\lambda(x) = f(x) . \tag{11}$$

THEOREM 2. *If* $f(x) \in L_2(E_n)$, *then there exists for sufficiently small* $|\lambda|$ *a unique generalized solution* $u_\lambda(x) \in W_2^{2m}(E_n)$ *of equation* (11).

Proof. We can deduce formally from equation (11), that:

$$(H + \lambda G)\nabla^{2m} u_\lambda(x) = f(x), \quad \text{or} \tag{12}$$

$$\nabla^{2m} u_\lambda(x) = (I + \lambda H^{-1} G)^{-1} H^{-1} f(x)$$

Since $f(x) \in L_2(E_n)$, then $(I + \lambda H^{-1} G)^{-1} H^{-1} f(x) \in L_2(E_n)$.

Consequently there exists a unique inverse operator $T = (\nabla^{2m})^{-1}$ in the space $L_2(E_n)$ such that

$$u_\lambda(x) = T(I + \lambda H^{-1} G)^{-1} H^{-1} f(x) \in W_2^{2m}(E_n).$$

The last formula represents the unique solution of equation (11) in the space $W_2^{2m}(E_n)$.

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