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Michele Caputo

Elasticity with dissipation represented by a simple memory mechanism

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Geofisica. — *Elasticity with dissipation represented by a simple memory mechanism* (*). Nota (**) del Corrisp. MICHELE CAPUTO.

RIASSUNTO. — Si introduce nelle relazioni fra sforzo e deformazione dell'elasticità un particolare meccanismo di memoria rappresentato da una derivata di ordine frazionario e si ottengono i periodi di oscillazione propria libera di una piastra e di uno strato sferico. Si ottiene anche il Q^{-1} e si dà un cenno di discussione del fenomeno di isteresi.

1. INTRODUCTION

One of the most classic problems of elasticity is that of giving a mathematical model of the elastic and dissipative properties of a medium, which has a Q^{-1} almost frequency independent. This problem was brought back to the only empirical part of the classical linear theory of elasticity by Caputo [1], [2], [3] and successfully tested by Caputo and Mainardi [4] and Mainardi [5], who introduced a derivative of real order in the stress-strain relation; Caputo [6] wrote it as follows:

$$\tau \sim \mu\varepsilon + \eta \frac{\partial^z \varepsilon}{\partial t^z}, \qquad 0 < z < 1 \tag{1}$$

where τ is the stress, ε the strain, μ, η and z represent the elastic and dissipative properties of the medium. By using (1), Caputo proved that also in linear elasticity one could have an almost frequency independent Q.

In (1) we have two parameters, η and z, to represent the dissipative or anelastic properties. A simpler, and probably physically more significant stress-strain relation, in which the dissipative properties are represented by one parameter only (namely z), is the following:

$$\tau \sim \eta \frac{\partial^z \varepsilon}{\partial t^z}, \qquad 0 < z < 1 . \tag{2}$$

Equation (2) reduces to the classic relation for $z = 0$. Here η has a different meaning than in (1); it gives the velocity of propagation of the signal in the medium, while in (1) it represented the non-elastic properties.

The new law is simpler and, as we shall see, represents well all the phenomena observed in linear elasticity; also it represents satisfactorily the anelastic properties with one parameter only.

Introducing (2) in the definition of strain and equilibrium condition, as done by Caputo [2] for (1), we obtain the equation which governs the motion in isotropic homogeneous media.

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For the infinite plate of finite thickness or the string, following the procedure of Caputo [2], we obtain:

$$\eta \frac{\partial^z}{\partial t^z} \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \tag{3}$$

where u is the displacement and x a coordinate perpendicular to the plate.

For the torsional motions of a spherical shell we obtain instead:

$$\frac{\partial^z}{\partial t^z}\left\{\eta \frac{1}{r} \frac{\partial^2 ru}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial \vartheta}\left[\frac{1}{\sin\vartheta} \frac{\partial u \sin\vartheta}{\partial \vartheta}\right] + \frac{\partial \eta}{\partial r} r \frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right\} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where u is the displacement, r and θ radius vector and colatitude; u is parallel to the surfaces r and θ constant of the point considered.

To obtain the free modes and the Q^{-1}, the boundary conditions are zero stress at the free surfaces in both cases.

2. The infinite plate

Let us consider the origin of the x coordinate on one side of the plate and let h be the thickness of the plate. Also let us assume $u = f(t) \cdot U(x)$.

Using the theorem of Appendix A of Caputo [6] for the Fourier transform of $f(t)$, we obtain from (3):

$$-\rho\omega^{2-z}\, U = \eta\varphi(z) \frac{\partial^2 U}{\partial x^2} \tag{5}$$

$$\varphi(z) = e^{i\frac{\pi z}{2}}.$$

If we consider the class of solutions

$$U = Ae^{i\alpha x} + Be^{-i\alpha x} \tag{6}$$

we obtain from (5):

$$\rho\omega^{2-z} = \eta\alpha^2\, \varphi(z) \tag{7}$$

where the parameter α is obtained from the boundary condition at $x = 0$ and $x = h$ for any t

$$\frac{\partial u}{\partial x} = 0. \tag{8}$$

Therefore

$$\begin{cases} A - B = 0 \\ Ae^{i\alpha h} - Be^{-i\alpha h} = 0 \end{cases} \tag{9}$$

$$\begin{cases} A = B \\ h\alpha = k\pi \quad , \quad k = \text{integer} \end{cases} \tag{10}$$

from which follows

$$\omega^{2-z} = \frac{\eta}{\rho}\,\varphi(z)\,\frac{k^2\pi^2}{h^2} \tag{11}$$

which gives the possible free frequencies.

Because of the dissipation ($z \neq 0$), for each value of k, we have more that one eigenfrequency.

3. The spherical shell (homogeneous)

In this case too we shall consider the class of solutions $U(r\theta)f(t)$; taking the Fourier transform of (4):

$$\varphi(z)\,\omega^z\left\{\eta\left\{\frac{1}{r}\,\frac{\partial^2 rU}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial}{\partial\vartheta}\left[\frac{1}{\sin\vartheta}\,\frac{dU\sin\vartheta}{d\vartheta}\right]\right\} + \right. \tag{12}$$

$$\left. + \frac{\partial\eta}{\partial r}\,r\,\frac{\partial}{\partial r}\left[r\,\frac{\partial}{\partial r}\left(\frac{U}{r}\right)\right]\right\} = -\rho\,\omega^2 U$$

and considering the class of solutions:

$$U = R(r)\,\frac{dP_n}{d\vartheta} \tag{13}$$

where $P_n(\theta)$ is the Legendre polynomial of order n, we obtain:

$$\frac{1}{r}\,\frac{d^2 R_r}{dr^2} + \frac{2}{r}\,\frac{dR}{dr} - \left[\frac{n(n+1)}{r^2} - \frac{\rho\omega^2}{\eta\omega^z\varphi(z)}\right] R = 0 \tag{14}$$

which gives

$$R = \left[A_n\, J_{n+1/2}\left(\sqrt{\frac{\rho\omega^2}{\eta\omega^z\varphi(z)}}\,r\right) + A_{-n}\, J_{-n-1/2}\left(\sqrt{\frac{\rho\omega^2}{\eta\omega^z\varphi(z)}}\,r\right)\right]\frac{1}{\sqrt{r}} \tag{15}$$

where $J_{\pm n \pm 1/2}$ are as usual the Bessel functions.

The free periods and Q^{-1} are obtained from the boundary condition:

$$\frac{\partial u/r}{\partial r} = 0 \tag{16}$$

for any t and θ at the outer and inner surfaces of the shell $r = R_1$, $r = R_2$ (e.g. see Caputo [2]).

4. A non-homogeneous spherical shell

Let us assume that $\rho = \bar{\rho}/r^2$. In this case (14) is

$$\frac{1}{r}\,\frac{d^2 rR}{dr^2} - \frac{R}{r^2}\left[n(n+1) - \frac{\bar{\rho}\omega^2}{\eta\omega^z\varphi(z)}\right] = 0 \tag{17}$$

whose solutions are

$$\begin{aligned} R &= A_{1,n}\, r^{m_1} + A_{2,n}\, r^{m_2} \\ m_{1,2} &= -\frac{1}{2} \pm \left[-\frac{\bar{\rho}\omega^2}{\eta\omega^z\varphi(z)} + \left(n+\frac{1}{2}\right)^2\right]^{1/2}. \end{aligned} \tag{18}$$

Using the same procedure of Caputo [2] we find that, to satisfy the boundary conditions, ω should satisfy the following equation

$$(19) \qquad (m_1 - 1)(m_2 - 1) \begin{vmatrix} R_2^{m_1} & R_2^{m_2} \\ R_1^{m_1} & R_1^{m_2} \end{vmatrix} = 0 .$$

The solutions of interest are given by

$$-\bar{\rho}\omega^2 + \eta\omega^z \varphi(z)[n^2 + n - 2] = 0$$

$$\omega = \left[\frac{\eta\varphi(z)(n^2 + n - 2)}{\bar{\rho}}\right]^{1/(2-z)}$$

which are independent of R_1 and R_2.

For a thin homogeneous shell of radius R_3 and density ρ we obtain for the free modes:

$$(21) \qquad \omega = \left[\frac{\eta\varphi(z)(n^2 + n - 2)}{\rho R_3^2}\right]^{1/(2-z)} .$$

One may easily verify that all the formulae above can be easily extended to the case $z < 0$, which is also of physical interest.

5. Hysteresis

For applications of engineering interest we may also estimate the hysteresis loop of the elastic materials which behave following the memory mechanism introduced with (2). By representing the strain with a saw tooth and computing the z order derivative appearing in (2) according to the definition given by Caputo [2] we obtained a hysteresis loop in agreement with the experimental results. It is found also:

$$Q^{-1} = \omega^z \sin\frac{\pi z}{2} .$$

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