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On the three-dimensional computation of geodetic networks and anomalies

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Geodesia. — *On the three-dimensional computation of geodetic networks and anomalies.* Nota (*) del Socio ANTONIO MARUSSI.

RIASSUNTO. — In una precedente pubblicazione (Marussi, 1973) sono stati posti i fondamenti della corrispondenza fra il campo potenziale attuale della gravità e quello normale di confronto, ambedue riferiti alle loro coordinate intrinseche; in questa Nota si studiano in dettaglio le proprietà geometriche della rappresentazione affine che viene così stabilita fra elementi infinitesimi del campo attuale e del campo immagine. Si danno in forma esplicita l'equazione fondamentale della geodesia fisica, le condizioni di Villarceau generalizzate nello spazio, e la condizione di armonicità del potenziale di disturbo.

LIST OF SYMBOLS

W	actual potential of Earth;
Φ , Λ	astronomical latitude and longitude;
$\boldsymbol{g} , g$	gravity vector and gravity;
U	normal potential;
φ , λ	normal or geodetic latitude and longitude;
$\boldsymbol{\gamma} , \gamma$	normal gravity vector and gravity;
T	anomalous potential;
$\tau = \frac{T}{U}$	relative anomaly in the potential;
$\Delta g = g - \gamma_a$	gravity anomaly;
$\mathfrak{z} = \frac{\Delta g}{\gamma}$	relative anomaly in gravity;
$\xi , \eta = \varepsilon \cos \varphi$	deflections of the vertical;
R	radius of curvature of line of force;
r	mean radius of Earth;
$\underset{P}{\gamma}_{ij}$	normal metric tensor evaluated at P;
$\boldsymbol{i}_r$	orthonormal geodetic triad in the field U ($\boldsymbol{i}_1$ Northw., $\boldsymbol{i}_2$ Eastw., $\boldsymbol{i}_3$ Zenith);
A , Z	astronomical azimuth and zenith distance;
α , z	normal or geodetic azimuth and zenith distance;
ρ , N	radius of curvature of meridian and of normal section perpendicular to it;
$f = \frac{\partial \lg \gamma}{\partial \varphi} = (\lg \gamma)_{/\varphi}$;	
ω	angular speed of rotation of Earth;
$T_{/r} = \frac{\partial T}{\partial x^r}$;	
δ^i_j	Kronecker's delta.

1. In some problems of Physical Geodesy, e.g. in Molodensky's problem, it is usual to compare measurable geometrical or physical quantities pertaining to the actual gravity field W of the Earth, with the homologous ones in a

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mathematically defined reference field U, e.g. the "normal" ellipsoidal field according to Somiglianza-Pizzetti's theory as adopted by the International Association of Geodesy, or the "Standard Earth" field as obtained from satellite Geodesy. The prerequisite for such comparison is the establishment of a one-to-one correspondence between points P of the field W and points Q of the field U. It is convenient to choose the point Q at an appropriate location on the equipotential surface $U = W^0$, where W^0 is the potential of the actual level surface $W = W^0$ on which P is located. The final choice of the point Q on $U = W^0$ is afterwards somewhat arbitrary, and can be best made by minimizing geometric distorsions in the mapping procedure. If point P belongs to the actual physical surface of the Earth, then point Q belongs by definition to the "telluroid" (Hirvonen, 1960; 1961).

After this preliminary mapping operation has been done, all metric properties of geometric figures can be obtained by using the metric tensor pertaining to the normal field. The procedure is therefore the generalisation in three dimensions of the one currently used in two dimensions in the conventional representation of the geoid onto the ellipsoid, and might be applied in the actual computation of triangulations and trilaterations in space, with the advantage of avoiding the separate treatement of planimetric and altimetric coordinates.

In the following, the properties of the correspondence are given for infitesimal elements attached to the corresponding points P and Q; the properties for finite elements as needed in the computation of geodetic networks can afterwards be obtained e.g. by generalizing Legendre's expansions, as shown in (Marussi, 1950).

The fundamental equation of physical geodesy, the generalized Villarceau conditions, and the harmonicity condition for the anomalous potential are also explicitly written.

2. Assume that the fields W and U are superposed in such a way that their centres of mass and their rotational axes coincide, and call $\varphi(P)$, $\lambda(P)$ and $U(P)$ (latitude, longitude, potential) the intrinsic "normal" or "geodetic" coordinates of P in the field U, whereas $\Phi(P)$, $\Lambda(P)$ and $W(P)$ the "astronomical" latitude and longitude, and the actual potential, are the intrinsic coordinates of the same point in the field W. We therefore have

$$\Phi = \varphi + \xi \quad , \quad \Lambda = \lambda + \varepsilon \quad , \quad W = U + T \tag{2.1}$$

in which ξ and $\varepsilon = \frac{\eta}{\cos\varphi}$ are by definition the anomalies in latitude and longitude, and T the anomalous potential.

As illustrated in a previous paper (Marussi, 1973), the correspondence between P and Q will be defined by the following equations

$$\varphi(Q) = \varphi(P) \quad , \quad \lambda(Q) = \lambda(P) \quad , \quad U(Q) = W(P) = U(P) + T(P) ; \tag{2.2}$$

such correspondence is therefore deflections- and potential- invariant, and it might be interpreted geometrically as the projection of point P onto point Q along the "isozenithal" line of the field U. The projection along the line of force as usually considered leads to more complicated relations than (2.2), but it should be noted that the difference in position resulting from the two procedures is very slight, being of the order of $\frac{T}{\gamma R} = \tau \frac{r}{R}$.

In the following the metric properties of the representation will be derived, and the relations between the adopted coordinates and measurable quantities will be given. We assume as measurables the geopotential W and g (as obtained by leveling and gravity observations), the elementary displacement $dP = \boldsymbol{t} ds$, where the unit vector $\boldsymbol{t}$ is observed by means of its actual astronomical azimuth A and zenith distance Z and ds by distance measuring instruments or triangulation procedures, and finally the astronomical latitude and longitude, Φ and Λ.

3. Indicating by $x^i (x^1 = \varphi, x^2 = \lambda, x^3 = U)$ the geodetic coordinates of P in the field U, and similarly by y^i the geodetic coordinates of Q, we have for a displacement $dP = \boldsymbol{t} ds$ and by (2.2) for its image $dQ = \boldsymbol{t}^* ds^*$ the following contravariant components:

$$dP \equiv dx^i = \lambda^i ds \quad ; \quad \lambda^i = \frac{dx^i}{ds} ; \tag{3.1'}$$

$$dQ \equiv dy^i = (\delta^i_r + \delta^i_3 T_{/r}) dx^r = \lambda^{i*} ds^* ; \tag{3.1''}$$

$$\lambda^{i*} = \frac{dy^i}{ds^*} = \frac{ds}{ds^*} (\delta^i_r + \delta^i_3 T_{/r}) \lambda^r \tag{3.1'''}$$

since $dW = dU + dT = dU + T_{/r} dx^r$, and the summation convention is used throughout.

Indicating by $\underset{P}{\gamma_{ij}}$ and $\underset{Q}{\gamma_{ij}}$ the components of the metric tensor of the "normal" field evaluated at P and at Q respectively, we then have

$$ds^2 = \overline{dP}^2 = \underset{P}{\gamma_{ij}} dx^i dx^j \quad ; \quad \underset{P}{\gamma_{ij}} \lambda^i \lambda^j = 1 ; \tag{3.2'}$$

$$ds^{*2} = \overline{dQ}^2 = \underset{Q}{\gamma_{ij}} dy^i dy^j \quad ; \quad \underset{Q}{\gamma_{ij}} \lambda^{i*} \lambda^{j*} = 1 ; \tag{3.2''}$$

$$m^2_\lambda = \left(\frac{ds^*}{ds}\right)^2 = a_{rs} \lambda^r \lambda^s \tag{3.2'''}$$

in which

$$a_{rs} = \underset{Q}{\gamma_{ij}} (\delta^i_r + \delta^i_3 T_{/r}) (\delta^j_s + \delta^j_3 T_{/s}) . \tag{3.3}$$

Let us now write

$$\boldsymbol{t} = \frac{dP}{ds} = \boldsymbol{i}_1 \cos \alpha \sin z + \boldsymbol{i}_2 \sin \alpha \sin z + \boldsymbol{i}_3 \cos z \tag{3.4}$$

where $\boldsymbol{i}_r$ is the orthonormal geodetic triad at P in the field U. By the definition of the correspondence between P and Q, such triad is identical with that at Q.

Here α and z are the " normal " or " geodetic " azimuth and zenith distance of the displacement dP; if

$$\text{(3.5)} \qquad \begin{aligned} A &= \alpha + \Delta\alpha \\ Z &= z + \Delta z \end{aligned}$$

are the measurable (astronomical) azimuth and zenith distance referred to the actual vertical, we have by straightforward algebra

$$\text{(3.6)} \qquad \begin{aligned} \Delta\alpha &= \varepsilon \sin\varphi + (\xi \sin A - \varepsilon \cos\varphi \cos A) \operatorname{ctg} Z \,, \\ \Delta z &= - \xi \cos A - \varepsilon \cos\varphi \sin A \end{aligned}$$

and therefore

$$\text{(3.4')} \qquad \begin{aligned} \boldsymbol{t} = \boldsymbol{i}_1 & (\cos A \sin Z + \xi \cos Z + \varepsilon \sin\varphi \sin A \sin Z) + \\ & + \boldsymbol{i}_2 [\sin A \sin Z + \varepsilon (\cos\varphi \cos Z - \sin\varphi \cos A \sin Z)] + \\ & + \boldsymbol{i}_3 (\cos Z - \xi \cos A \sin Z - \varepsilon \cos\varphi \sin A \sin Z) \,. \end{aligned}$$

4. From this point on, we shall assume the normal ellipsoidal field as the U reference field; we have (Marussi, 1950)

$$\text{(4.1)} \qquad \gamma_{ij} = \begin{Vmatrix} \rho^2 & 0 & \dfrac{\rho f}{\gamma} \\ 0 & N^2 \cos^2\varphi & 0 \\ \dfrac{\rho f}{\gamma} & 0 & \dfrac{1+f^2}{\gamma^2} \end{Vmatrix}$$

and for the non-zero Christoffel symbols of the second kind:

$$\text{(4.2)} \qquad \begin{aligned} &\left\{ {1 \atop 11} \right\} = \left(\lg \frac{\rho}{\gamma}\right)_{/\varphi} ; \quad \left\{ {1 \atop 22} \right\} = \frac{N \cos^2\varphi}{\rho} (\operatorname{tg}\varphi - f) \; ; \quad \left\{ {2 \atop 12} \right\} = - \frac{\rho}{N} \operatorname{tg}\varphi \, ; \\ &\left\{ {1 \atop 13} \right\} = (\lg \rho)_{/U} \cong \frac{-1}{U} \, ; \left\{ {2 \atop 23} \right\} = (\lg N)_{/U} \cong \frac{-1}{U} \, ; \left\{ {1 \atop 33} \right\} = \frac{1}{\gamma\rho} f_{/U} \cong \frac{1}{U} f_{/U} \, ; \\ &\left\{ {3 \atop 11} \right\} = \gamma\rho \cong U \; ; \; \left\{ {3 \atop 22} \right\} = \gamma N \cos^2\varphi \cong U \cos^2\varphi \; ; \; \left\{ {3 \atop 33} \right\} = -(\lg \gamma)_{/U} \cong \frac{-2}{U} \,. \end{aligned}$$

In deriving the expressions of Christoffel's symbols, use has been made of the following relations:

$$\text{(4.3)} \qquad \begin{aligned} &(\lg \rho)_{/U} = \frac{-1}{\gamma\rho} (1 + f^2 - f_{/\varphi}) \cong \frac{-1}{U} \, ; \, (\lg N)_{/U} = \frac{-1}{\gamma N} (1 + f \operatorname{tg}\varphi) \cong \frac{-1}{U} \; ; \\ &(\lg N)_{/\varphi} = \left(1 - \frac{\rho}{N}\right) \operatorname{tg}\varphi \;\; ; \;\; (\lg \gamma)_{/U} = \frac{1}{\gamma} \left(\frac{1}{N} + \frac{1+f^2}{\rho} + \frac{2\,\omega^2}{\gamma} \right) \cong \frac{2}{U} \,. \end{aligned}$$

It should be noted that consequently, ignoring terms of higher order, the components of the metric tensor $\underset{P}{\gamma_{ij}}$ at P are expressed in terms of the components of $\underset{Q}{\gamma_{ij}}$ at Q by the following relations

$$(4.4)\qquad \begin{aligned} \underset{P}{\gamma_{11}} &= \underset{Q}{\gamma_{11}}(1+2\tau) \quad ; \quad \underset{P}{\gamma_{22}} = \underset{Q}{\gamma_{22}}(1+2\tau)\,; \\ \underset{P}{\gamma_{13}} &= \underset{Q}{\gamma_{13}}(1+3\tau) \quad ; \quad \underset{P}{\gamma_{33}} = \underset{Q}{\gamma_{33}}(1+4\tau) \end{aligned}$$

since the contravariant components of the displacement vector $Q - P$ are $\Delta x^i = \delta_3^i T$.

The contravariant components $\lambda^i = \frac{dx^i}{ds}$ of $\boldsymbol{t}$ at P are then given by the following formulae:

$$(4.5)\qquad \begin{aligned} \lambda^1 &= \frac{d\varphi}{ds} = \frac{l^1}{\rho_P}\,, \\ \lambda^2 &= \frac{d\lambda}{ds} = \frac{l^2}{N_P \cos\varphi}\,, \\ \lambda^3 &= \frac{dU}{ds} = -\gamma_P\, l^3 \end{aligned}$$

in which the non-dimensional coefficients l^i are given by

$$(4.5')\qquad \begin{aligned} l^1 &= \cos A \sin Z + f \cos Z + \xi(\cos Z - f \cos A \sin Z) + \\ &\quad + \varepsilon \sin A \sin Z (\sin\varphi - f\cos\varphi) \\ l^2 &= \sin A \sin Z + \varepsilon(\cos\varphi \cos Z - \sin\varphi \cos A \sin Z) \\ l^3 &= \cos Z - \xi \cos A \sin Z - \varepsilon \cos\varphi \sin A \sin Z\,. \end{aligned}$$

Substituting in (3.2‴) we then have

$$(4.6)\qquad a_{rs} \equiv \left\| \begin{matrix} \rho^2 & 0 & \frac{\rho(f-\xi)}{\gamma} \\ 0 & N^2\cos^2\varphi & \frac{-N\eta}{\gamma} \\ \frac{\rho(f-\xi)}{\gamma} & \frac{-N\eta}{\gamma} & \frac{1+2\zeta+4\tau}{\gamma^2} \end{matrix} \right\|$$

since (Marussi, 1973)

$$(4.7)\qquad T_{/\varphi} = -\gamma\rho\xi \cong -U\xi \;,\quad T_{/\lambda} = -\gamma N\eta \cong -U\eta \;,\quad T_{/U} = \zeta + 2\tau$$

and

$$(4.7')\qquad m^2 = \left(\frac{ds^*}{ds}\right)^2 = 1 - 2\tau\sin^2 Z + 2\zeta\cos^2 Z + \\ + 2[(\xi - f)\cos A + \varepsilon \sin A]\sin Z \cos Z$$

$$(4.7'')\qquad m = \frac{ds^*}{ds} = 1 - \tau \sin^2 Z + \zeta\cos^2 Z + [(\xi - f)\cos A + \varepsilon\sin A]\sin Z\cos Z.$$

It might be noted that in the correspondence horizontal displacements are contracted in the ratio $(1-\tau)$, and isozenithal displacements are expanded in the ratio $(1 + \zeta)$. The horizontal contraction corresponds to the projection of the geoid onto the ellipsoid in the classical approach, and the vertical dilatation to the adoption of normal heights.

5. Let us now consider the vector of gravity $\boldsymbol{g} = -g\boldsymbol{i}_3$; we have by definition

$$(5.1') \qquad \boldsymbol{g} = -\underset{Q}{\gamma}\,[\xi\boldsymbol{i}_1 + \eta\boldsymbol{i}_2 + (1 + \zeta)\,\boldsymbol{i}_3]\,;$$

$$(5.1'') \qquad \Delta\boldsymbol{g} = \boldsymbol{g} - \underset{Q}{\boldsymbol{\gamma}} = -\underset{Q}{\gamma}\,(\xi\boldsymbol{i}_1 + \eta\boldsymbol{i}_2 + \zeta\boldsymbol{i}_3)\,;$$

$$(5.1''') \qquad \Delta g = g - \underset{Q}{\gamma} = \underset{Q}{\gamma}\,\zeta \quad ; \quad g = (1 + \zeta)\,\underset{Q}{\gamma}$$

since $\underset{Q}{\boldsymbol{\gamma}} = -\underset{Q}{\gamma}\boldsymbol{i}_3$ because of the deflection-invariance of the correspondence between P and Q.

It has been shown in a previous paper (Marussi, 1973) that the following relations hold, here written in their most concise form:

$$(5.2) \qquad \zeta = \frac{\partial T}{\partial U} - T\frac{\partial \lg\gamma}{\partial U} + T\left(\frac{1}{\gamma\rho}\frac{\partial f}{\partial U}\frac{\partial T}{\partial\varphi} - \frac{\partial \lg\gamma}{\partial U}\frac{\partial T}{\partial U}\right) + f\xi$$

$$(5.3) \qquad \begin{aligned} &\frac{\partial\gamma\,\rho\xi}{\partial\lambda} = \frac{\partial\gamma\,N\eta\cos\varphi}{\partial\varphi} \\ &\frac{\partial(\zeta - f\xi)}{\cos\varphi\,\partial\lambda} = -\gamma\frac{\partial N\eta}{\partial U} \\ &\frac{\partial(\zeta - f\xi)}{\partial\varphi} = -\gamma\frac{\partial\rho\xi}{\partial U} - T\frac{\partial f}{\partial U}\,. \end{aligned}$$

With the approximations indicated in (4.3), the preceeding formulae might also be written as follows

$$(5.2') \qquad \frac{\partial T}{\partial U} = \zeta + 2\tau \qquad \text{or} \qquad \zeta = \gamma\frac{\partial}{\partial U}\left(\frac{T}{\gamma}\right)$$

$$(5.3') \qquad \begin{aligned} &\frac{\partial\xi}{\partial\lambda} = \frac{\partial\eta\cos\varphi}{\partial\varphi} \\ &\frac{\partial\zeta}{\partial\varphi} = -U^2\frac{\partial}{\partial U}\left(\frac{\xi}{U}\right) \\ &\frac{\partial\zeta}{\cos\varphi\,\partial\lambda} = -U^2\frac{\partial}{\partial U}\left(\frac{\eta}{U}\right)\,. \end{aligned}$$

Formulae (5.2) and (5.2') express the fundamental equation of physical geodesy; the first of formulae (5.3) and (5.3') gives the condition first established by Villarceau for the existence of the equipotential surfaces, and the last two generalize such existence condition to the equi-latitudinal and the equi-longitudinal surfaces.

In a private communication, prof. T. Krarup has recalled my attention on the fact the harmonicity of T should also be accounted for; Laplace's equation

$$(5.4) \qquad \Delta_2 T = \frac{1}{\sqrt{\Gamma}} \frac{\partial}{\partial x^i} (\sqrt{\Gamma}\, \gamma^{ik} T_{/k}) = 0 \quad , \quad \Gamma = |\gamma_{ij}|$$

is easily written in explicit form, and leads with the usual approximations to the following condition

$$(5.5) \qquad \frac{\partial \xi \cos\varphi}{\partial\varphi} + \frac{\partial\varepsilon}{\partial\lambda} = U \cos\varphi \frac{\partial(\zeta + 2\tau)}{\partial U} + \frac{2\rho\omega^2}{\gamma}(\zeta + 2\tau)\cos\varphi\,.$$

Ignoring the last term, and noting that $\zeta + 2\tau = \frac{\partial T}{\partial U}$, we also have

$$(5.5') \qquad \frac{\partial \xi \cos\varphi}{\partial\varphi} + \frac{\partial\varepsilon}{\partial\lambda} = U \cos\varphi \frac{\partial^2 T}{\partial U^2}\,.$$

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