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On the second boundary value problem in the linear ştheory of micropolar elasticity

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Meccanica. — *On the second boundary value problem in the linear theory of micropolar elasticity.* Nota di D. IEŞAN, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — In questa Nota si stabilisce un teorema di esistenza per il secondo problema fondamentale nella teoria di elasticità micropolare.

In this paper we consider the second boundary value problem in the static theory of anisotropic and inhomogeneous micropolar elastic solids. Using the method given in [1] we establish an existence theorem.

Let us consider a finite region Ω of three dimensional space, bounded by the surface S. In what follows we denote the components of the displacement vector by u_i, the components of the microrotation vector by φ_i, the components of the body force vector by f_i and the components of the body couple vector by l_i.

The field equations of the static theory of micropolar elasticity can be written in the form [2]

$$\text{(1)} \qquad \mathrm{A}u = f\,,$$

where

$$\text{(2)} \qquad u = (u_i\,,\,\varphi_i) \quad , \quad f = (f_i\,;\,l_i)\,,$$

and Au has the components

$$\text{(3)} \qquad \mathrm{A}_i\, u = -\frac{\partial}{\partial x_j}\left(\mathrm{A}_{jikl}\frac{\partial u_l}{\partial x_k} + \mathrm{B}_{jikl}\frac{\partial \varphi_l}{\partial x_k} + \mathrm{A}_{jikl}\,\varepsilon_{lkm}\,\varphi_m\right),$$

$$\mathrm{A}_{3+i}\, u = -\frac{\partial}{\partial x_j}\left(\mathrm{B}_{klji}\frac{\partial u_l}{\partial x_k} + \mathrm{C}_{jikl}\frac{\partial \varphi_l}{\partial x_k} + \mathrm{B}_{klji}\,\varepsilon_{lkm}\,\varphi_m\right) -$$

$$- \varepsilon_{ijk}\,\mathrm{A}_{jkrs}\frac{\partial u_s}{\partial x_r} - \varepsilon_{ijk}\,\mathrm{B}_{jkrs}\frac{\partial \varphi_s}{\partial x_r} - \varepsilon_{ijk}\,\varepsilon_{srm}\,\mathrm{A}_{jkrs}\,\varphi_m\,.$$

In (3), ε_{ijk} is the alternating symbol and $\mathrm{A}_{ijkl}(x)$, $\mathrm{B}_{ijkl}(x)$, $\mathrm{C}_{ijkl}(x)$ are the constitutive coefficients.

We will consider the second boundary value problem of the micropolar elasticity theory, defined by the boundary conditions

$$\text{(4)} \qquad p(u) = 0 \quad \text{on} \quad \mathrm{S}\,,$$

(*) Nella seduta del 26 novembre 1973.

where

$$p(u) = (t_i(u), m_i(u)), \tag{5}$$

$$t_i(u) = \left(A_{jikl}\frac{\partial u_l}{\partial x_k} + B_{jikl}\frac{\partial \varphi_l}{\partial x_k} + A_{jikl}\,\varepsilon_{lkm}\,\varphi_m\right) n_j ,$$

$$m_i(u) = \left(B_{klji}\frac{\partial u_l}{\partial x_k} + C_{jikl}\frac{\partial \varphi_l}{\partial x_k} + B_{klji}\,\varepsilon_{lkm}\,\varphi_m\right) n_j ,$$

n_j being the direction cosines of the outward normal to S at x.

In [2] we established the following relation

$$\int_\Omega u\,\mathrm{A}u\,\mathrm{d}x = -\int_S up(u)\,\mathrm{d}x + 2\int_\Omega \mathrm{U}(u)\,\mathrm{d}x , \tag{6}$$

where U is the internal energy density

$$2\,\mathrm{U}(u) = \mathrm{A}_{ijkl}\,e_{ij}\,e_{kl} + 2\,\mathrm{B}_{ijkl}\,e_{ij}\,\varkappa_{kl} + \mathrm{C}_{ijkl}\,\varkappa_{ij}\,\varkappa_{kl} . \tag{7}$$

In (7) we have used the notations

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\,\varphi_k \quad , \quad \varkappa_{ij} = \varphi_{j,i} . \tag{8}$$

In our case the relation (6) becomes

$$\int_\Omega u\,\mathrm{A}u\,\mathrm{d}x = 2\int_\Omega \mathrm{U}(u)\,\mathrm{d}x . \tag{9}$$

It is known (see e.g. [2]) that the solution of the boundary value problem (1), (4) is determined to within an additive vector of the form

$$u_i^0 = a_i + \varepsilon_{ijk}\,b_j\,x_k \quad , \quad \varphi_i^0 = b_i , \tag{10}$$

where a_i and b_i are arbitrary constants.

In what follows we establish an existence theorem using the results from [1]. The functions $\mathrm{A}_{ijkl}, \mathrm{B}_{ijkl}, \mathrm{C}_{ijkl}$ are supposed to belong to C^∞ and the quadratic form U is supposed to be positive definite, so that

$$2\,\mathrm{U}(u) \geq c \sum_{i,j=1}^{3}{}' (e_{ij}^2 + \varkappa_{ij}^2) , \tag{11}$$

where c is a positive constant. We assume that Ω is C^∞-smooth [1, p. 61].

Taking into account (8) we can write

$$e_{ij} = \gamma_{ij} + \varepsilon_{ijk}(r_k - \varphi_k) , \tag{12}$$

where

$$2\,\gamma_{ij} = u_{i,j} + u_{j,i} \quad , \quad r_k = \frac{1}{2}\,\varepsilon_{knm}\,u_{m,n}. \tag{13}$$

We have

$$\sum_{i,j=1}^{3} e_{ij}^2 = \sum_{i,j=1}^{3} \{\gamma_{ij}^2 + 2\,\gamma_{ij}\,\varepsilon_{ijk}(r_k - \varphi_k) + [\varepsilon_{ijk}(r_k - \varphi_k)]^2\} = \tag{14}$$

$$= \sum_{i,j=1}^{3} \{\gamma_{ij}^2 + [\varepsilon_{ijk}(r_k - \varphi_k)]^2\} \geq \sum_{i,j=1}^{3} \gamma_{ij}^2 ,$$

so that

$$2\,\mathrm{U}(u) \geq c \sum_{i,j=1}^{3} (\gamma_{ij}^2 + \varkappa_{ij}^2) . \tag{15}$$

To prove the existence theorem for the boundary value problem (1), (4), as in [1, p. 91], we consider the equation

$$\mathrm{A}u + q_0\,u = f , \tag{16}$$

where q_0 is any positive constant. First we consider the boundary value problem (16), (4). Using (9) it follows that the inequality to be proven in this case is the following ([1, p. 62])

$$2\int_\Omega \mathrm{U}(u)\,\mathrm{d}x + q_0 \int_\Omega u^2\,\mathrm{d}x \geq c_0\,\|u\|_1^2 ; \qquad c_0 > 0 \quad (c_0 = \text{const.}) , \tag{17}$$

for any $u = (u_i , \varphi_i) \in \mathrm{H}_1(\Omega)$. By $\mathrm{H}_1(\Omega)$ is denoted the Hilbert function space obtained by the functional completition of $\mathrm{C}^1(\bar{\Omega})$ with respect to the scalar product

$$(u , v)_1 = \int_\Omega \mathrm{D}^s u\,\mathrm{D}^s v\,\mathrm{d}x , \qquad (0 \leq s \leq 1) . \tag{18}$$

Using (15) it is easily seen that the inequality (17) is implied by the following inequality

$$\int_\Omega \sum_{i,j=1}^{3} (\gamma_{ij}^2 + \varkappa_{ij}^2)\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x \geq c_1\,\|u\|_1^2 ; \quad c_1 > 0 , \ (c_1 = \text{const.}) . \tag{19}$$

In what follows we shall prove the inequality (19). We denote

$$u^{(1)} = (u_1 , u_2 , u_3 , 0 , 0 , 0) \quad , \quad u^{(2)} = (0 , 0 , 0 , \varphi_1 , \varphi_2 , \varphi_3) .$$

Using the second Korn's inequality [1, p. 92], we obtain

$$\int_\Omega \sum_{i,j=1}^{3} \gamma_{ij}^2\,\mathrm{d}x + \int_\Omega [u^{(1)}]^2\,\mathrm{d}x \geq c_2\,\|u^{(1)}\|_1^2 ; \quad c_2 > 0 , \ (c_2 = \text{const.}) . \tag{20}$$

Obviously, we have

$$\int_\Omega \sum_{i,j=1}^{3} (\varphi_{i,j})^2\,\mathrm{d}x + \int_\Omega [u^{(2)}]^2\,\mathrm{d}x = \|u^{(2)}\|_1^2 . \tag{21}$$

Using (20) and (21) it follows the inequality (19) which implies (17). From (17) it follows that the boundary value problem (16), (4) has only one solution which is C^∞ in $\overline{\Omega}$. Since our differential system is formally self-adjoint, it follows that a C^∞ solution of the following equation

$$\mathrm{A}u + q_0\, u - \lambda u = f\,, \tag{22}$$

with the boundary condition (4), exists when and only when

$$\int_\Omega f\rho \,\mathrm{d}x = 0\,, \tag{23}$$

where ρ is any C^∞ solution of the boundary value problem (22), (4) with $f = 0$. In the case $\lambda = q_0$ the only C^∞ solution of the homogeneous system is u^0 given by (10). From (23), (10) it follows

THEOREM. *The boundary value problem* (1), (4) *has solutions belonging to* $C^\infty(\overline{\Omega})$ *if and only if the* C^∞ *function f satisfies the condition*

$$\int_\Omega f u^0 \,\mathrm{d}x = 0\,, \tag{24}$$

where u^0 *is given by* (10).

Taking into account (2), (10) and the fact the constants a_i and b_i are arbitrary, the relation (24) implies and is implied by the following relations

$$\int_\Omega f_i \,\mathrm{d}x = 0 \quad , \quad \int_\Omega (\varepsilon_{ijk}\, x_j f_k + l_i)\,\mathrm{d}x = 0\,. \tag{25}$$

These relations mean that the resultant force and the total moment should be zero.

The solutions of the boundary value problem (1), (4) given by above Theorem minimize the functional [1]

$$\Phi(u) = \int_\Omega [\mathrm{U}(u) - uf]\,\mathrm{d}x\,, \tag{26}$$

in $H_1(\Omega)$.

Other existence theorems in micropolar elasticity were established in [3-6].

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