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The initial value problem for the neutron transport equation by the semi-group method

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Fisica matematica. — *The initial value problem for the neutron transport equation by the semi-group method* (*). Nota di ENZO BARONE, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Si studiano due problemi differenziali che si presentano nella teoria di trasporto dei neutroni in uno spazio di Sobolev dissimmetrico con peso con il metodo dei Semi-gruppi di operatori e si stabilisce l'esistenza e unicità della soluzione.

§ 1. INTRODUCTION

In neutron transport theory [2], the following differential problems are of great interest:

$$(1.2)\quad \begin{cases} \dfrac{\partial u}{\partial t} = -\mu \cdot \dfrac{\partial u}{\partial x} + \dfrac{c}{2}\displaystyle\int_{-1}^{1} u(x\,,\mu'\,;t)\,\mathrm{d}\mu' & \text{where } c > 0 \\ u(-a\,,\mu\,;t) = 0 & \text{if } 0 < \mu \leq 1,\ t > 0 \\ u(a\,,\mu\,;t) = 0 & \text{if } -1 \leq \mu < 0,\ t > 0 \\ u(x\,,\mu\,;0) = u_0(x\,,\mu) \end{cases}$$

$$(1.2)\quad \begin{cases} \dfrac{\partial u}{\partial t} = -\mu \dfrac{\partial u}{\partial x} + \dfrac{c}{2}\displaystyle\int_{-1}^{1} u(x\,,\mu'\,;t)\,\mathrm{d}\mu' & \text{where } c > 0 \\ u(-a\,,\mu\,;t) = u(a\,,\mu\,;t) & \text{if } |\mu| \leq 1,\ t > 0 \\ u(x\,,\mu\,;0) = u_0(x\,,\mu) \end{cases}$$

where $u(x\,,\mu\,;t)$ is defined in $\mathrm{I}\times\mathbf{R}_+$, $\mathrm{I} = [-a\,,a]\times[-1\,,1]$ and $\mathbf{R}_+$ is the set of real numbers ≥ 0. The unknown $u(x\,,\mu\,;t)$ is related to the neutron density $\mathrm{N}(x\,,\mu\,;t')$ (note [5], pp. 217) by the transformation:

$$\mathrm{N}(x\,,\mu\,;t') = \exp(-\sigma t)\cdot u(x\,,\mu\,;t)\,, \qquad t = vt'.$$

Problems (1.1) and (1.2) may be written more synthetically as follows

$$(1.3)\quad \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \mathrm{A}u \\ u(0) = u_0 \end{cases}$$

(*) Work performed under the auspices of the C.N.R.
(**) Nella seduta del 26 novembre 1973.

provided that we consider u as transformation from $\mathbf{R}_+$ into H, where H is an appropriate functional space and A is an operator, whose domain D (A) conglobates the boundary conditions appearing in (1.1) and (1.2).

There is a wide literature about the study of two problems synthesized by (1.3) where $H = L_2(I)$ (note [5], [6]; for further references see note [1]) by Semi-groups theory of boundedlinear operators (note [3]). But as far as we know, system (1.3) has never been studied by the Sobolev spaces theory (note [7]). This seems connected with the fact that, wanting to use Semi-groups theory, it is impossible to face the problem if $H = W_2^1(I)$: in fact, since the trace theorem holds in this case, D (A) cannot be dense in $W_2^1(I)$ (note Hille-Yosida theorem, [3]). It nevertheless seems fairly natural to investigate the problem in a suitable unsymmetric Sobolev space $W_*(I)$, as we will specify in the next number.

In two papers that are going to be published in "Ricerche di Matematica", we proved in detail that, in the $W_*(I)$ space, the operators A (of each of the two problems) are infinitesimal generators of (C_0) Semi-groups (note [3], [4], [8]). Therefore, for a result of Semi-groups theory, there exists and is unique the solution of both (1.1) and (1.2) for any choice of u_0 in D(A). More precisely, we proved that the operator A of the 1st problem is a $G(1, c)$ operator (note [4]), while the one of the 2nd problem is a $G(M, 1/a + Mc)$ operator, where $M > 1$.

The reason of the choice of the space $W_*(I)$ lies in the greater regularity of the solution in such a space. Precisely, the solution belongs to $W_2^1(I_1)$, for each closed set I_1', contained in I, without points in common with the straight line $\mu = 0$ (note in § 2 D(B) = D(A) definition).

§ 2. Notation and Results

Let $\mathcal{E}(I)$ then be the set of the functions with complex values and of class C^∞ in I. In $\mathcal{E}(I)$, we define the following scalar product:

$$(f \mid g)_* = (f \mid g) + (\mu f_\mu \mid \mu g_\mu) \qquad \forall f, g \in \mathcal{E}(I) \tag{2.1}$$

where $(. \mid .)$ is the usual scalar product in $L_2(I)$, i e.,

$$(f \mid g) = \int_I f \bar{g} .$$

Let $W_*(I)$ then be the completion of $\mathcal{E}(I)$ with respect to $(\cdot \mid \cdot)_*$; obviously $W_*(I)$ is a Hilbert space.

Let us consider following operators:

$$\begin{cases} B_* \, . = -\mu \dfrac{\partial}{\partial x} \, . \\ D(B_*) = \{ f \in \mathcal{E}(I) \mid f(-a, \mu) = 0 \\ \text{if } \mu \in]0, 1] \text{ and } f(a, \mu) = 0 \text{ if } \mu \in [-1, 0[\} \end{cases} \tag{2.2}$$

in 1^{st} problem,

$$(2.2)' \quad \begin{cases} B_* \cdot = -\mu \dfrac{\partial}{\partial x} \cdot \\ D(B_*) = \{ f \in \mathcal{E}(I) \mid f(-a, \mu) = f(a, \mu) \text{ where } |\mu| \leq 1 \} \end{cases}$$

in 2^{nd} problem, and in both problems

$$(2.3) \quad \begin{cases} J_* \cdot = \dfrac{1}{2} \displaystyle\int_{-1}^{1} \cdot \, d\mu' \\ D(J_*) = \mathcal{E}(I). \end{cases}$$

It is evident that *with said definitions* B_* and J_* are linear operators in $\mathcal{E}(I)$ consequently in $W_*(I)$.

Operator B_* is not bounded in $W_*(I)$ but it may be closed. Then we can define B as smallest closed extension.

On the contrary, J_* is bounded with norm 1 and therefore we indicate by J its extension to $W_*(I)$

Then, we define operator A as:

$$(2.4) \qquad A = B + cJ.$$

Evidently: $D(A) = D(B)$, since $D(J) = W_*(I)$.

Operator A may be considered as a perturbation of B by the bounded operator cJ. Hence, it is enough to prove that B is an infinitesimal generator of a (C_0) Semi-group (note [3], pp. 389).

More precisely, I proved the following theorems:

THEOREM 1. *If* $I' =]-a, a[\times [-1, 1]$ *and*

$$\mathfrak{D}(I') = \{ h \in \mathcal{E}(I) \mid \exists \eta > 0 \ni' h(x, \mu) = 0 \quad \forall x \in [-a, -a+\eta] \cup [a-\eta, a] \}$$

$\mathfrak{D}(I)$ *is dense in* $W_*(I)$. *Consequently,* $D(B)$ (*in both cases*) *is dense in* $W_*(I)$ *because* $\mathfrak{D}(I') \subset D(B_*) \subset D(B)$.

THEOREM 2. *In the* 1^{st} *problem, we have*:

$$\forall \lambda > 0 : \lambda \in \rho(B) \wedge \| R(\lambda; B) \| < \frac{1}{\lambda} ;$$

therefore $B \in G(1, 0)$.

THEOREM 3. *In the second problem, we have*:

$$(2.5) \qquad \forall \lambda > 0 : \lambda \in \rho(B), \quad \forall \lambda > \frac{1}{a} : \| R^n(\lambda; B) \| < \frac{M}{\left(\lambda - \frac{1}{a}\right)^n}$$

for each positive integer n, *where* M *is a constant* >1. *Therefore,*

$$B \in G\left(M, \frac{1}{a}\right).$$

§ 3. Summary of the proofs of the main theorems

THEOREM 1. *First, we note that, since* $\mathcal{E}(\mathrm{I})$ *is dense in* $\mathrm{W}_*(\mathrm{I})$, *it is enough to prove that* $\mathcal{D}(\mathrm{I}')$ *is dense in* $\mathcal{E}(\mathrm{I})$ *i.e.*

$$(3.1) \qquad \forall f \in \mathcal{E}(\mathrm{I}) \quad \textit{and} \quad \forall \varepsilon > 0 \quad \exists h \in \mathcal{D}(\mathrm{I}') \ni' \|f - h\|_* < \mathrm{K}\varepsilon$$

where K *is a positive constant, independent of* ε.

For this purpose, fixed f and ε, we define:

$$g(x,\mu) = \begin{cases} f(x,\mu) & \text{where } x \in [-a+2\varepsilon, a-2\varepsilon], \ |\mu| \le 1 \\ 0 & \text{where } x \in [-a, -a+\varepsilon] \cup [a-\varepsilon, a], \ |\mu| \le 1 \\ \dfrac{x+a-\varepsilon}{\varepsilon} f(-a+2\varepsilon, \mu) & \text{where } x \in [-a+\varepsilon, -a+2\varepsilon], \ |\mu| \le 1 \\ \dfrac{-x+a-\varepsilon}{\varepsilon} f(a-2\varepsilon, \mu) & \text{where } x \in [a-2, a-\varepsilon], \ |\mu| \le 1 \end{cases}$$

and we prove $\|f - g\|_*^2 \le \mathrm{K}_1 \varepsilon$ if K_1 is a constant > 0.

Then, we consider the functions $g_\lambda(x,\mu) = g(\lambda x, \lambda\mu)$.

$\forall (x,\mu) \in \mathbf{R}^2 \ni' (\lambda x, \lambda\mu) \in \mathrm{I}$ where $\lambda \in]0, 1[$ and we observe that they are the same kind of g and that they are defined on

$$\mathrm{I}_\lambda = \left[-\frac{a}{\lambda}, \frac{a}{\lambda}\right] \times \left[-\frac{1}{\lambda}, \frac{1}{\lambda}\right] \supset \mathrm{I}.$$

We then prove (note also [7], Chapt. 2, § 3)

$$\exists \lambda' \in \left]1 - \frac{\varepsilon}{a}, 1\right[\ni' \|g_{\lambda'} - g\|_* < \varepsilon.$$

Finally, we denote by $g_{\lambda',\eta}$, where $\eta > 0$, the mean (averaged) function for $g_{\lambda'}$.

These are of class C^∞ and such that $g_{\lambda',\eta}(x,\mu) = 0$

$$\forall x < -\frac{a+\varepsilon}{\lambda'} - \eta \quad , \quad \forall x > \frac{a-\varepsilon}{\lambda'} + \eta .$$

Among these, we may find a $g_{\lambda',\eta'}$ where $\eta' \in \left]0, a - \frac{a-\varepsilon}{\lambda}\right[$ such that $\|g_{\lambda',\eta'} - g_{\lambda'}\|_* < \varepsilon$. This $h = g_{\lambda',\eta'}$ function belongs to $\mathcal{D}(\mathrm{I}')$ and verifies the (3.1) where $\mathrm{K} = \mathrm{K}' + 2$.

THEOREM 2. *Let be* $\lambda > 0$. *In order to prove that* $\lambda \in \rho(\mathrm{B})$, *as* $\mathcal{D}(\mathrm{I}')$ *is dense in* $\mathrm{W}_*(\mathrm{I})$, *it is enough to show that,* $\forall g \in \mathcal{D}(\mathrm{I}')$, *the equation (in* f*)*:

$$(3.2) \qquad (\lambda \mathrm{I} - \mathrm{B}) f = g$$

admits a unique $f \in \mathrm{D}(\mathrm{B})$ *solution.*

Then, in order to prove that R (λ ; B) is a bounded operator with norm $< 1/\lambda$, we have to show that

$$\|f\|_* < \frac{1}{\lambda}\|g\|_*, \qquad \text{where } f = R(\lambda ; B) g. \tag{3.3}$$

The bounded operator R (λ ; B) is defined on $\mathfrak{D}(I')$ and it may be prolonged over the whole $W_*(I)$. Consequently, (3.3) will be valid $\forall g \in W_*(I)$.

We begin studying the problem:

$$\begin{cases} \lambda f(x, \mu) + \mu \frac{\partial}{\partial x} f(x, \mu) = g(x, \mu) & \\ f(-a, \mu) = 0 & \mu \in]0, 1] \\ f(a, \mu) = 0 & \mu \in [-1, 0[\end{cases} \tag{3.4}$$

in $I'' = [-a, a] \times ([-1, 0[\cup]0, 1])$ if $g \in \mathfrak{D}(I')$ and $\lambda > 0$.

We obtain the solution

$$f(x, \mu) = \int_{\pm a}^{x} \frac{\exp\left[-\frac{\lambda}{\mu}(x-t)\right]}{\mu} g(t, \mu)\, dt \tag{3.5}$$

where we take the + if $\mu < 0$ and the − if $\mu > 0$.

Then, defining $f(x, 0) = \frac{1}{\lambda} g(x, 0)$, we prove that $f \in \mathcal{E}(I)$ by induction.

Thus, problem (3.4) and equation (3.2) coincide and, therefore, we have only to prove (3.3).

We notice that, if $\mu \neq 0$:

$$f(x, \mu) = \int_{-\infty}^{+\infty} K(x-t, \mu) \cdot g_0(t, \mu)\, dt = K * g_0$$

where g_0 is the g prolonged to zero if $x < -a$, $x > a$ and

$$K(z, \mu) = \begin{cases} 0 & z < 0, \quad \mu > 0 \\ \frac{1}{\mu}\exp\left(-\frac{\lambda}{\mu} z\right) & z > 0, \quad \mu > 0 \\ \frac{-1}{\mu}\exp\left(-\frac{\lambda}{\mu} z\right) & z < 0, \quad \mu < 0 \\ 0 & z > 0, \quad \mu < 0. \end{cases}$$

Denoting by $\hat{g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(ixt)\, g(t)\, dt$ the Fourier transformation (of the g), as $K(\cdot, \mu)$ and $g_0(\cdot, \mu)$ are some functions that may be added in **R**, we have:

$$\hat{K}(x, \mu) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\lambda - ix}$$

then, by using Parseval equality, we get

$$\int_{-a}^{a} | f(x , \mu) |^2 \, dx \leq \int_{-\infty}^{+\infty} | K * g_0 |^2 \, dx = \int_{-\infty}^{+\infty} | \hat{K} \cdot \hat{g}_0 |^2 \, dx \leq$$

$$\leq \frac{1}{2 \pi \lambda^2} \int_{-\infty}^{+\infty} | \hat{g}_0 |^2 \, dx = \frac{1}{2 \pi \lambda^2} \int_{-a}^{a} | g(x , \mu) |^2 \, dx$$

then

$$\| f \| \leq \frac{1}{\sqrt{2 \pi}} \cdot \frac{1}{\lambda} \| g \| . \tag{3.6}$$

In order to evaluate $\| \mu f_\mu \|$, notice that, by deriving with respect to μ and by multiplying (3.4) by μ, we have:

$$\begin{cases} \lambda (\mu f_\mu) + \mu \dfrac{\partial}{\partial x} (\mu f_\mu) = \mu g_\mu + \lambda f - g \\ \mu f_\mu (-a , \mu) = 0 & \mu > 0 \\ \mu f_\mu (a , \mu) = 0 & \mu < 0 \end{cases}$$

which means that μf_μ has the form (3.5), where, instead of g, there is $\mu g_\mu + \lambda f - g$. By the same argument used to obtain (3.6), it follows:

$$\| \mu f_\mu \| \leq \frac{1}{\sqrt{2 \pi}} \cdot \frac{1}{\lambda} \| \mu g_\mu + \lambda f - g \| \leq \frac{1}{\sqrt{2 \pi}} \cdot \frac{1}{\lambda} \left[\| \mu g_\mu \|^2 + \left(\frac{1}{\sqrt{2 \pi}} + 1 \right) \| g \| \right] .$$

Hence our assertions are completely proved.

THEOREM 3. *As for Theorem 2, we first prove that, if* $\lambda > 0$ *and* $g \in \mathfrak{D}(I')$, *then the* (3.2) *functional equation is equivalent to the problem:*

$$\begin{cases} \lambda f(x , \mu) + \mu \dfrac{\partial}{\partial x} f(x , \mu) = g(x , \mu) \\ f(-a , \mu) = f(a , \mu) \end{cases} \tag{3.7}$$

by showing that its solution:

$$f(x , \mu) = \begin{cases} \dfrac{1}{\exp\left(\dfrac{2 a \lambda}{\mu}\right) - 1} \cdot \left[\displaystyle\int_{-a}^{x} \frac{\exp \frac{\lambda}{\mu} [2 a - (x - t)]}{\mu} g(t , \mu) \, dt + \right. \\ \qquad \left. + \displaystyle\int_{x}^{a} \frac{\exp \left[\frac{-\lambda}{\mu} (x - t) \right]}{\mu} g(t , \mu) \, dt \right] & \mu \neq 0 \\ \dfrac{1}{\lambda} g(x , 0) \qquad \text{where } \mu = 0 \end{cases} \tag{3.8}$$

is of C^∞ *class, by induction.*

Secondly we prove inequality (2.5) as follows.
We put:

$$(3.9) \qquad K(z,\mu) = \begin{cases} 0 & \text{if } Z < -2a \quad \text{or } Z > 2a \\[2mm] \dfrac{1}{\exp\left(\dfrac{2a\lambda}{\mu}\right)-1} \, \dfrac{\exp\left[\dfrac{\lambda}{\mu}(2a-Z)\right]}{\mu} & \text{if } Z \in [0, 2a] \\[2mm] \dfrac{1}{\exp\left(\dfrac{2a\lambda}{\mu}\right)-1} \, \dfrac{\exp\left(-\dfrac{\lambda}{\mu}Z\right)}{\mu} & \text{if } Z \in [-2a, 0] \end{cases}$$

then we put $g = 0$ if $x < -a$ and if $x > a$. As before, we then have that $f = K * g$ and by calculating the Fourier transformation of K, we get:

$$\hat{K}(x,\mu) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\lambda - ix\mu} \cdot \left[(1 + \cos 2ax) + i(\operatorname{sen} 2ax) \frac{1 + \exp \dfrac{2a\lambda}{\mu}}{1 - \exp \dfrac{2a\lambda}{\mu}} \right].$$

On the other hand, if $\lambda > 1/a$ we then have $\forall \mu \in [-1, 1] - \{0\}$

$$\left(\frac{1 + \exp \dfrac{2a\lambda}{\mu}}{1 - \exp \dfrac{2a\lambda}{\mu}} \right)^2 \le 2 \qquad \text{and, consequently,} \quad |\hat{K}|^2 \le \frac{2}{\pi} \frac{1}{\lambda^2}.$$

By the same method used in Theorem 2, we prove that:

$$\| f \|_*^2 = \| f \|^2 + \| \mu f_\mu \|^2 \le \frac{2}{\pi} \cdot \left[\frac{6}{\pi} + 4 \right] \cdot \frac{1}{\lambda^2} \cdot \| g \|_*^2.$$

To prove the assertion fully, it is enough to show (note [3], p. 37) that:

$$(3.10) \quad \exists M > 0, \ \forall \lambda > \frac{1}{a} \quad \text{and} \quad \forall n \in \mathbf{N}: \left\| \frac{\partial^n f}{\partial \lambda^n} \right\|_*^2 \le \frac{M^2 (n!)^2}{\lambda^{2(n+1)}} \cdot \| g \|_*^2.$$

For this purpose, by deriving (3.7) n times with respect to λ we have by Leibnitz rule:

$$(3.11) \quad \begin{cases} \lambda \dfrac{\partial^n f}{\partial \lambda^n} + \mu \dfrac{\partial}{\partial x} \dfrac{\partial^n f}{\partial \lambda^n} = -n \dfrac{\partial^{n-1} f}{\partial \lambda^{n-1}} \\[2mm] \left(\dfrac{\partial^n f}{\partial \lambda^n} \right)(-a, \mu) = \left(\dfrac{\partial^n f}{\partial \lambda^n} \right)(a, \mu). \end{cases}$$

Then, just as the solution of (3.7) may be put into the form $f = K * g$, $\dfrac{\partial^n f}{\partial \lambda^n}$, solution of a problem similar to the (3.7), where instead of g there is $-n \dfrac{\partial \lambda^{n-1} f}{\partial \lambda^{n-1}}$, may be written as follows

$$\frac{\partial^n f}{\partial \lambda^n} = -K * \left(n \frac{\partial^{n-1} f}{\partial \lambda^{n-1}} \right) \qquad \forall n \in \mathbf{N}.$$

Thus, for recursion and for Parseval equality, we have

$$\left\| \frac{\partial^n f}{\partial \lambda^n} \right\|^2 = (n!)^2 \cdot \| \hat{K}^{n+1} \cdot \hat{g} \|^2 \leq (n!)^2 \cdot \left(\frac{2}{\pi}\right)^{n+1} \cdot \frac{1}{\lambda^{2(n+1)}} \cdot \| g \|^2 .$$

Again, by deriving with respect to μ and by multiplying by μ we have easily from (3.11):

$$\left\| \mu \frac{\partial}{\partial \mu} \frac{\partial^n f}{\partial \lambda^n} \right\|^2 \leq [(n+1)!]^2 \cdot \left(\frac{2}{\pi}\right)^{n+1} \cdot \frac{3}{\lambda^{2(n+1)}} \left(\frac{2}{\pi} + 1\right) [\| g \|^2 + \| \mu g_\mu \|^2] .$$

Then:

$$\left\| \frac{\partial^n f}{\partial \lambda^n} \right\|_*^2 \leq (n!)^2 c' \cdot \frac{(n+1)^2}{\left(\frac{\pi}{2}\right)^{n+1}} \frac{1}{\lambda^{2(n+1)}} \cdot \| g \|_*^2 \qquad \text{where } c' = 6\left(\frac{2}{\pi} + 1\right)$$

that leads to the (3.10) since the sequence $\left(\frac{(n+1)^2}{\left(\frac{\pi}{2}\right)^{n+1}}\right)_n$ is infinitesimal hence bounded.

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