Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A note on semi-continuous mappings

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **55** (1973), n.5, p. 400–403.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_55_5_400_0>

Topologia. — *A note on semi-continuous mappings.* Nota di
TAKASHI NOIRI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Se X ed Y sono spazi topologici, un'applicazione $f: X \to Y$ dicesi *semicontinua* [3] se, per ogni insieme aperto V di $Y, f^{-1}(V)$ risulta un insieme semiaperto di X. Qui si studiano tali applicazioni e si dànno tre loro caratterizzazioni.

1. INTRODUCTION

In 1963, N. Levine [3] has defined a subset A of a topological space X to be *semi-open* if there exists an open set U of X such that $U \subset A \subset \mathrm{Cl}(U)$, where Cl (U) denotes the closure of U in X. He has also defined a mapping f of a topological space X into a topological space Y to be *semi-continuous* if for each open set V of $Y, f^{-1}(V)$ is a semi-open set of X. In 1970, N. Biswas [1] has given several characterizations of semi-continuous mappings. The purpose of the present Note is to give three characterizations of semi-continuous mappings and to investigate some properties of such mappings.

Throughout the present Note, X and Y will always denote topological spaces on which no separation axioms are assumed unless explicity stated. By $f: X \to Y$ we denote a mapping (not necessarily continuous) f of a topological space X into a topological space Y. When A is a subset of a topological space, we denote the closure of A and the interior of A by Cl (A) and Int (A), respectively. Furthermore, we shall denote the family of all semi-open sets in X by SO (X).

2. CHARACTERIZATIONS

DEFINITION 1. A subset A of X is said to be *semi-closed* if there exists a closed set F of X such that $\mathrm{Int}(F) \subset A \subset F$ [1].

LEMMA 1 (Biswas [1]). *Let* A *be a subset of* X. *Then, the following are equivalent*:

(1) A *is a semi-closed set of* X;
(2) $X - A \in \mathrm{SO}(X)$;
(3) $\mathrm{Int}(\mathrm{Cl}(A)) \subset A$.

THEOREM 1. *Let* $f: X \to Y$ *be a mapping. Then, the following are equivalent*:

(1) f *is semi-continuous*;
(2) $\mathrm{Int}(\mathrm{Cl}(f^{-1}(B))) \subset f^{-1}(\mathrm{Cl}(B))$ *for each subset* B *of* Y;
(3) $f(\mathrm{Int}(\mathrm{Cl}(A))) \subset \mathrm{Cl}(f(A))$ *for each subset* A *of* X.

(*) Nella seduta del 26 novembre 1973.

Proof. (1) $\Rightarrow$ (2): Suppose that f is semi-continuous, and let B be an arbitrary subset of Y. Then $f^{-1}(\mathrm{Cl}(\mathrm{B}))$ is a semi-closed set of X. Since we have always $\mathrm{Cl}(\mathrm{B}) \supset \mathrm{B}$, by Lemma 1, we obtain $f^{-1}(\mathrm{Cl}(\mathrm{B})) \supset \supset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{B})))) \supset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{B})))$.

(2) $\Rightarrow$ (3): Suppose that (2) holds, and let A be an arbitrary subset of X. Let us put $\mathrm{B} = f(\mathrm{A})$, then $\mathrm{A} \subset f^{-1}(\mathrm{B})$. Therefore, by hypothesis, we have $\mathrm{Int}(\mathrm{Cl}(\mathrm{A})) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{B}))) \subset f^{-1}(\mathrm{Cl}(\mathrm{B}))$. Consequently, we obtain $f(\mathrm{Int}(\mathrm{Cl}(\mathrm{A}))) \subset \mathrm{Cl}(f(\mathrm{A}))$.

(3) $\Rightarrow$ (1): Suppose that (3) holds, and let F be an arbitrary closed set of Y. Let us put $\mathrm{A} = f^{-1}(\mathrm{F})$, then $f(\mathrm{A}) \subset \mathrm{F}$. Therefore, by hypothesis, we have $f(\mathrm{Int}(\mathrm{Cl}(\mathrm{A}))) \subset \mathrm{Cl}(f(\mathrm{A})) \subset \mathrm{Cl}(\mathrm{F}) = \mathrm{F}$. Since we have always $f^{-1}(f(\mathrm{E})) \supset \mathrm{E}$ for each subset E of X, we obtain $f^{-1}(\mathrm{F}) \supset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{F})))$. Therefore, by Lemma 1, $f^{-1}(\mathrm{F})$ is a semi-closed set of X. This implies that f is semi-continuous.

LEMMA 2 (Levine [3]). *A mapping $f : \mathrm{X} \to \mathrm{Y}$ is semi-continuous if and only if for any point $x \in \mathrm{X}$ and any open set* V *of* Y *containing $f(x)$, there exists* $\mathrm{U} \in \mathrm{SO}(\mathrm{X})$ *such that $x \in \mathrm{U}$ and $f(\mathrm{U}) \subset \mathrm{V}$.*

THEOREM 2. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a mapping and let $g : \mathrm{X} \to \mathrm{X} \times \mathrm{Y}$, given by $g(x) = (x, f(x))$, be the graph mapping of f. Then, f is semi-continuous if and only if g is semi-continuous.*

Proof. Necessity. Suppose that f is semi-continuous. Let x be an arbitrary point of X and W be any open set of $\mathrm{X} \times \mathrm{Y}$ containing $g(x) = (x, f(x))$. Then there exist open sets $\mathrm{G} \subset \mathrm{X}$ and $\mathrm{V} \subset \mathrm{Y}$ such that $(x, f(x)) \in \mathrm{G} \times \mathrm{V} \subset \mathrm{W}$. Since f is semi-continuous, by Lemma 2, there exists $\mathrm{U}_0 \in \mathrm{SO}(\mathrm{X})$ such that $x \in \mathrm{U}_0$ and $f(\mathrm{U}_0) \subset \mathrm{V}$. Let us put $\mathrm{U} = \mathrm{U}_0 \cap \mathrm{G}$, then we have $x \in \mathrm{U} \in \mathrm{SO}(\mathrm{X})$ [4, Lemma 1]. Furthermore, we have $g(\mathrm{U}) \subset \mathrm{G} \times \mathrm{V} \subset \mathrm{W}$. Therefore, by Lemma 2, we see that g is semi-continuous.

Sufficiency. Suppose that g is semi-continuous. Let x be an arbitrary point of X and V any open set of Y containing $f(x)$. Then $\mathrm{X} \times \mathrm{V}$ is an open set of $\mathrm{X} \times \mathrm{Y}$ containing $g(x)$. Since g is semi-continuous, by Lemma 2, there exists $\mathrm{U} \in \mathrm{SO}(\mathrm{X})$ such that $x \in \mathrm{U}$ and $g(\mathrm{U}) \subset \mathrm{X} \times \mathrm{V}$. Therefore, we obtain $f(\mathrm{U}) \subset \mathrm{V}$. Hence it follows from Lemma 2 that f is semi-continuous.

3. HAUSDORFF SPACES AND SEMI-CONTINUOUS MAPPINGS

Let $f : \mathrm{X} \to \mathrm{Y}$ be a mapping. The subset $\{(x, f(x)) \mid x \in \mathrm{X}\}$ of the product space $\mathrm{X} \times \mathrm{Y}$ is called the *graph* of f and is denoted by $\mathrm{G}(f)$. It is well known that $\mathrm{G}(f)$ is a closed set of $\mathrm{X} \times \mathrm{Y}$ whenever f is continuous and Y is Hausdorff. The following theorem is a modification of this fact.

THEOREM 3. *If $f : \mathrm{X} \to \mathrm{Y}$ is semi-continuous and* Y *is Hausdorff, then* $\mathrm{G}(f)$ *is a semi-closed set of* $\mathrm{X} \times \mathrm{Y}$.

Proof. Let (x, y) be an arbitrary point of $X \times Y - G(f)$. Then we have $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets W and V such that $f(x) \in W$ and $y \in V$. Moreover, by Lemma 2, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset W$ because f is semi-continuous. Therefore, we obtain $(x, y) \in U \times V \subset X \times Y - G(f)$. Since V is an open set of Y and $U \in SO(X)$, we have $U \times V \in SO(X \times Y)$ [3, Theor. 11]. Hence we obtain $X \times Y - G(f) \in SO(X \times Y)$ [3, Theor. 2]. Thus, it follows from Lemma 1 that $G(f)$ is a semi-closed set of $X \times Y$.

THEOREM 4. *If* $f: X \to Y$ *is semi-continuous and* Y *is Hausdorff, then the set* $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ *is semi-closed in the product space* $X \times X$.

Proof. Let Δ be the diagonal of $Y \times Y$. Since Y is Hausdorff, Δ is a closed set of $Y \times Y$, as is well known. Since f is semi-continuous, the product mapping $f \times f: X \times X \to Y \times Y$ is also semi-continuous [3, Theor. 14]. Therefore, $(f \times f)^{-1}(\Delta)$ is a semi-closed set of $X \times X$. It follows from a simple calculation that $(f \times f)^{-1}(\Delta) = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$. This completes the proof.

DEFINITION 2. The intersection of all semi-closed sets containing a set A is called the *semi-closure* of A and is denoted by $C_S(A)$ [1].

Remark. A set A is semi-closed if and only if $C_S(A) = A$.

DEFINITION 3. Let x be a point of X and N be a subset of X. N is called a *semi-neighborhood* of x in X if there exists $U \in SO(X)$ such that $x \in U \subset N$ [2].

LEMMA 3. *Let* A *be a subset of* X. *Then,* $x \in C_S(A)$ *if and only if for any semi-neighborhood* Nx *of* x *in* X, $A \cap Nx \neq \varnothing$.

Proof. Necessity. Suppose that $x \in C_S(A)$. If there exists a semi-neighborhood N of x in X such that $N \cap A = \varnothing$, then there exists $U \in SO(X)$ such that $x \in U \subset N$. Therefore, we have $U \cap A = \varnothing$, so that $A \subset X - U$. By Lemma 1, $X - U$ is semi-closed and hence $C_S(A) \subset X - U$. Since $x \notin X - U$, we obtain $x \notin C_S(A)$ which is contrary to the hypothesis.

Sufficiency. Suppose that every semi-neighborhood of x in X meets A. If $x \notin C_S(A)$, then there exists a semi-closed set F of X such that $A \subset F$ and $x \notin F$. Therefore, by Lemma 1, we have $x \in X - F \in SO(X)$. Hence $X - F$ is a semi-neighborhood of x in X, but $(X - F) \cap A = \varnothing$. This is contrary to the hypothesis.

DEFINITION 4. Let A be a subset of X. A mapping $\gamma: X \to A$ is called a *semi-continuous retraction* if γ is semi-continuous and the restriction $\gamma \mid A$ is the identity mapping on A.

THEOREM 5. *Let* A *be a subset of* X *and* $\gamma: X \to A$ *be a semi-continuous retraction. If* X *is Hausdorff, then* A *is a semi-closed set of* X.

Proof. Suppose that A is not semi-closed. Then there exists a point x in X such that $x \in C_S(A)$ but $x \notin A$. It follows that $\gamma(x) \neq x$ because γ is a semi-continuous retraction. Since X is Hausdorff, there exist disjoint open sets U and V in X such that $x \in U$ and $\gamma(x) \in V$. Now let W be an arbitrary semi-neighborhood of x. Then $W \cap U$ is a semi-neighborhood of x [4, Lemma 1]. Since $x \in C_S(A)$, by Lemma 3, we have $(W \cap U) \cap A \neq \varnothing$. Therefore, there exists a point y in $W \cap U \cap A$. Since $y \in A$, we have $\gamma(y) = y \in U$ and hence $\gamma(y) \notin V$. This implies that $\gamma(W) \not\subset V$ because $y \in W$. This is contrary to the semi-continuity of γ. Consequently, A is a semi-closed set of X.

REFERENCES

[1] N. BISWAS, *On characterizations of semi-continuous functions*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. », (8) *48*, 399–402 (1970).

[2] E. BOHN and J. LEE, *Semi-topological groups*, « Amer. Math. Monthly », *72*, 996–998 (1965).

[3] N. LEVINE, *Semi-open sets and semi-continuity in topological spaces*, « Amer. Math. Monthly », *70*, 36–41 (1963).

[4] T. NOIRI, *On semi-continuous mappings*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. » (to appear).