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Union curves and union curvature of a curve in special Kawaguchi spaces of order two

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Geometria differenziale. — *Union curves and union curvature of a curve in special Kawaguchi spaces of order two.* Nota di UDAI PRATAP SINGH e SHRI KRISHNA DEO DUBEY, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Diversi Autori hanno già studiato le *union curves* (*curve assiali*) e la relativa curvatura sugli spazi di Finsler. In questo lavoro tale teoria viene estesa ad uno speciale spazio di Kawaguchi di dimensione pari. È anche ottenuta l'espressione della curvatura geodetica delle « union curves ».

1. INTRODUCTION

In an n-dimensional special Kawaguchi space Kn of order 2, the arc length of the curve $x^i = x^i(t)$ (1) is given by

$$S = \int [A_i(x, \overset{\prime}{x}) \overset{\prime\prime}{x}{}^i + B(x, \overset{\prime}{x})]^{1/p} \, dt , \qquad p \neq 0, 3/2 \tag{1.1}$$

where

$$\overset{\prime}{x}{}^i = \frac{dx^i}{dt} \quad , \quad \overset{\prime\prime}{x}{}^i = \frac{d^2 x^i}{dt^2}$$

and A_i , B are differentiable functions of x^i and $\overset{\prime}{x}{}^i$.

In order that the arc length in the space should remain unaltered under any transformation of the parameter t, we must have the so-called Zermelo conditions,

$$A_i \overset{\prime}{x}{}^i = 0 \quad , \quad A_{i(j)} \overset{\prime}{x}{}^j = (p - 2) A_i \quad , \quad B_{(i)} \overset{\prime}{x}{}^i = pB , \tag{1.2}$$

where

$$A_{i(j)} = \frac{\partial A_i}{\partial \overset{\prime}{x}{}^j} \quad , \quad B_{(i)} = \frac{\partial B}{\partial \overset{\prime}{x}{}^i} .$$

And since s given by (1.1) is a scalar, it follows that A_i is a vector. The notations and symbols employed (without explanations) in this paper are same as those of A. Kawaguchi [1] (2).

We consider an m-dimensional subspace Km of Kn represented by the equations $x^i = x^i(u^\alpha)$ and the matrix of the projection factors $p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ has rank m. If we denote a_α and b the quantities in Km corresponding to A_i and B in Kn respectively, it follows that the equations similar to (1.2) hold for a_α and b. Putting

$$G_{ij} = 2 A_{i(j)} - A_{j(i)} \quad , \quad G_{\alpha\beta} = 2 a_{\alpha(\beta)} - a_{\beta(\alpha)}$$

(*) Nella seduta del 26 novembre 1973.

(1) Latin indices run from 1 to n, Greek ones $\alpha , \beta , \gamma , \delta , \varepsilon , \rho$ from 1 to m and μ , ν from $m + 1$ to n.

(2) Numbers in the brackets refer to the references at the end of the paper.

it follows (Yohida [3]),

$$G_{ij}\, p^i_\alpha\, p^j_\beta = G_{\alpha\beta}\,. \tag{1.3}$$

On assumption that n, m are both even and det (G_{ij}) does not vanish identically, it is easy to show that

$$p^\alpha_i\, p^i_\beta = \delta^\alpha_\beta \quad \text{and} \quad G^{\beta\alpha}\, G_{\beta\gamma} = \delta^\alpha_\gamma\,, \tag{1.4}$$

where $p^\alpha_i \overset{\text{def}}{=} G^{\alpha\beta}\, G_{ij}\, p^j_\beta$ ($G^{\alpha\beta}$ being the tensor reciprocal to $G_{\alpha\beta}$).

The connection Γ^i of Kn and Γ^α of Km are given as

$$2\,\Gamma^i = (2\,A_{lm}\, x'^m - B_{(l)})\, G^{li} \quad , \quad 2\,\Gamma^\alpha = (2\,a_{\beta\gamma}\, u'^\gamma - b_{(\beta)})\, G^{\beta\alpha} \tag{1.5}$$

where

$$A_{lm} = \frac{\partial A_l}{\partial x^m} \quad \text{and} \quad a_{\beta\gamma} = \frac{\partial a_\beta}{\partial u^\gamma}\,.$$

The covariant differential of a contravariant vector field $v^i(x^i, x'^i)$ homogeneous of degree zero with respect to x'^i is defined by Kawaguchi [1],

$$\delta v^i = dv^i + \Gamma^i_{jK}\, v^j\, dx^K \tag{1.6}$$

where

$$\Gamma^i_{jK} = \frac{\partial^2 \Gamma^i}{\partial x'^j\, \partial x'^K} = \Gamma^i_{Kj}\,.$$

If v^α is a vector field in Km such that $v^i = p^i_\alpha\, v^\alpha$ then the induced covariant differential $\check{\delta} v^\alpha (= p^\alpha_i\, \delta v^i)$ is given by

$$\check{\delta} v^\alpha = dv^\alpha + \check{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, du^\gamma, \tag{1.7}$$

and it has been shown that ([3])

$$\check{\Gamma}^\alpha_{\beta\gamma} = p^\alpha_i\,(p^i_{\beta\gamma} + \Gamma^i_{jK}\, p^j_\beta\, p^K_\gamma)\,. \tag{1.8}$$

Yoshida ([3]) has defined

$$\overset{\circ}{H}{}^i_{\beta\alpha} \overset{\text{def}}{=} \overset{\circ}{D}_\beta\, p^i_\alpha \overset{\text{def}}{=} p^i_{\alpha\beta} + \Gamma^i_{jK}\, p^j_\beta\, p^K_\alpha - \check{\Gamma}^\gamma_{\alpha\beta}\, p^i_\gamma, \tag{1.9}$$

where

$$p^i_{\alpha\beta} = \partial p^i_\alpha / \partial u^\beta\,.$$

Using the vectors $\underset{\mu}{n}{}^i$ of Kn, normal to Km, $\overset{\circ}{H}{}^i_{\beta\alpha}$ can be expressed as

$$\overset{\circ}{H}{}^i_{\beta\alpha} = H^\mu_{\beta\alpha}\, \underset{\mu}{n}{}^i \tag{1.10}$$

where

$$H^\mu_{\beta\alpha} = G^{\nu\mu}\, G_{ij}\, \underset{\nu}{n}{}^i\, \overset{\circ}{H}{}^j_{\beta\alpha} = G^{\mu\nu}\, \underset{\nu}{n}{}_i\, \overset{\circ}{H}{}^i_{\beta\alpha}\,. \tag{1.11}$$

For the tangent vector dx^i/ds of a curve $C : x^i = x^i(s)$ of the subspace K_m, Yoshida ([3]) has obtained the relation

$$\frac{\delta \dot{x}^i}{ds} - p^i_\alpha \frac{\check{\delta} \dot{u}^\alpha}{ds} = \overset{\circ}{H}{}^i_{\beta\gamma} \dot{u}^\beta \dot{u}^\gamma \tag{1.12}$$

where

$$\dot{x}^i = \frac{dx^i}{ds} \quad \text{and} \quad \dot{u}^\alpha = \frac{du^\alpha}{ds} \cdot$$

2. UNION CURVES

E. Bompiani was possibly the first person to consider, more than half a century ago, the union curves on a surface of a projective 3–dimensional space under the name of axial symmetric curves. These notions have been extended to Riemannian and Finsler spaces. We now proceed to discuss union curves in special Kawaguchi spaces.

Let $C : x^i = x^i(s)$ be a curve of K_m. Consider a congruence of curves in K_n given by a vector field λ^i such that to each point of K_n there passes one and only one curve of the congruence. At the points of the subspace K_m, we may write ([4])

$$\lambda^i = t^\alpha p^i_\alpha + \sum_\nu \Gamma_{(\nu)} \underset{\nu}{n}{}^i \quad , \quad \Gamma_{(\nu)} \neq 0. \tag{2.1}$$

It is assumed that this vector is normalised by the condition (as in Yoshida [3])

$$G_{ij} \lambda^i \lambda^j = 1 . \tag{2.2}$$

At a point of a curve C, the linear space generated by the vectors $\frac{dx^i}{ds}\left(= p^i_\alpha \frac{du^\alpha}{ds}\right)$ and $q^i\left(= \frac{\delta}{ds}(\dot{x}^i)\right)$ is called the osculating surface of the curve. The curve is said to be union curve relative to the above mentioned congruence if its osculating variety contains the vector λ^i. Thus, for union curve relative to λ^i, we have,

$$\lambda^i = A p^i_\alpha \frac{du^\alpha}{ds} + B q^i \tag{2.3}$$

where A and B are parameters to be determined. Now, the equations (1.10), (1.12), (2.1) and (2.3) yield

$$t^\alpha p^i_\alpha + \sum_\nu \Gamma_{(\nu)} \underset{\nu}{n}{}^i = \left(A \frac{du^\alpha}{ds} + B p^\alpha\right) p^i_\alpha + B H^\nu_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \underset{\nu}{n}{}^i \tag{2.4}$$

where

$$p^\alpha = \frac{\check{\delta} \dot{u}^\alpha}{ds} \cdot$$

Since p^i_α and $\underset{\nu}{n}{}^i$ are linearly independent, so we have,

$$t^\alpha = A \frac{du^\alpha}{ds} + B p^\alpha , \tag{2.5}$$

and

$$\Gamma_{(\nu)} = \mathrm{B} \mathrm{H}^{\nu}_{\alpha\beta} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} . \tag{2.6}$$

In the case of an even-dimensional special Kawaguchi space of order two, the following conditions hold (Watanabe [2]),

$$\delta \mathrm{A}_{i(j)} = 0 \quad , \quad \mathrm{A}_{i(j)(\mathrm{K})} = 0 \quad \text{and} \quad \mathrm{G}_{ij(\mathrm{K})} = 0 . \tag{2.7}$$

Applying these conditions, we obtain,

$$\frac{\delta \mathrm{G}_{ij}}{ds} = 0 \quad \text{and} \quad \mathrm{G}_{\alpha\beta}\, p^{\alpha} \frac{du^{\beta}}{ds} = 0 . \tag{2.8}$$

The equations (2.3) and (2.8) give

$$\lambda \overset{\text{def}}{=} \mathrm{G}_{ij} \frac{dx^{i}}{ds} \lambda^{j} = \mathrm{A} . \tag{2.9}$$

Putting the values of A and B in the equation (2.5), we find

$$p^{\alpha} = \left(\mathrm{H}^{\nu}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} / \Gamma_{(\nu)} \right) \left(t^{\alpha} - \lambda \frac{du^{\alpha}}{ds} \right) . \tag{2.10}$$

The equations (2.10) are the differential equations of the union curves provided that the ratio

$$\mathrm{H}^{\nu}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} / \Gamma_{(\nu)}$$

is independent of ν.

From (2.9), we get

$$\sum_{\nu} \Gamma^{2}_{(\nu)}\, p^{\alpha} p^{\rho} = k_{n} \left(t^{\alpha} - \lambda \frac{du^{\alpha}}{ds} \right) \left(t^{\rho} - \lambda \frac{du^{\rho}}{ds} \right) \tag{2.11}$$

where

$$k_{n} \overset{\text{def}}{=} \sum_{\nu} \mathrm{H}^{\nu}_{\gamma\beta} \mathrm{H}^{\nu}_{\delta\varepsilon} \frac{du^{\gamma}}{ds} \frac{du^{\beta}}{ds} \frac{du^{\delta}}{ds} \frac{du^{\varepsilon}}{ds} \tag{2.12}$$

is defined as normal curvature of the subspace (Yoshida [3]). Hence, we conclude

THEOREM 2.1. *If a union curve is geodesic in the subspace of an even-dimensional special Kawaguchi space* ($n > 2$) *and the normal curvature of* Km *in the direction of the curve is not zero, then the component of the congruence tangential to the subspace is tangent to the curve i.e.*

$$t^{\alpha} = \lambda \frac{du^{\alpha}}{ds} .$$

THEOREM. 2.2. *If the congruence of a curve is normal to the subspace* Km, *the union curve is a geodesic on the subspace of an even-dimensional special Kawaguchi space.*

Proof. Since the congruence is normal (i.e. $t^\alpha = 0$), the equation (2.1) yields

$$\lambda = G_{ij} \frac{dx^i}{ds} \lambda^j = 0 .$$

The equation (2.10) reduces to $p^\alpha = 0$. This completes the proof.

3. UNION CURVATURE

The vector with contravariant components

$$\eta^\alpha \overset{\text{def}}{=} p^\alpha - \left\{ \sum_\nu \left(H^\nu_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\nu \Gamma^2_{(\nu)} \right\}^{1/2} \left(t^\alpha - \lambda \frac{du^\alpha}{ds} \right), \tag{3.1}$$

is called the union curvature vector of C relative to a congruence λ^i and its magnitude k_u is called the union curvature of the curve. From (3.1), we have

$$k_u^2 = G_{\alpha\rho} \eta^\alpha \eta^\rho , \tag{3.2}$$

whence

$$\begin{aligned} k_u^2 = k_g^2 - 2 \left\{ \sum_\nu \left(H^\nu_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\nu \Gamma^2_{(\nu)} \right\}^{1/2} \times \\ \times G_{\alpha\rho} t^\alpha p^\rho + \left\{ \sum_\nu \left(H^\nu_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\nu \Gamma^2_{(\nu)} \right\} \times \\ \times \left(t^\alpha - \lambda \frac{du^\alpha}{ds} \right) \left(t^\rho - \lambda \frac{du^\rho}{ds} \right) G_{\alpha\rho} \end{aligned} \tag{3.3}$$

where we have used the relation

$$G_{\alpha\rho} \frac{du^\alpha}{ds} p^\rho = 0$$

and

$$k_g^2 \overset{\text{def}}{=} G_{\alpha\beta} p^\alpha p^\beta . \tag{3.4}$$

The scalar k_g is called the geodesic curvature of the curve. If we define

$$t_\beta \overset{\text{def}}{=} G_{\alpha\beta} t^\alpha = G_{ij} \lambda^i p^j_\beta \tag{3.5}$$

then using (2.9) and $G_{\alpha\rho} \frac{du^\alpha}{ds} \frac{du^\rho}{ds} = 1$, we have

$$G_{\alpha\rho} \left(t^\alpha - \lambda \frac{du^\alpha}{ds} \right) \left(t^\rho - \lambda \frac{du^\rho}{ds} \right) = G_{\alpha\rho} t^\alpha t^\rho - \lambda^2 . \tag{3.6}$$

Thus the equation (3.3) reduces to

$$\begin{aligned} k_u^2 = k_g^2 - 2 \left\{ \sum_\nu \left(H^\nu_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\nu \Gamma^2_{(\nu)} \right\}^{1/2} \times \\ \times t_\rho p^\rho + \left\{ \sum_\nu \left(H^\nu_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right)^2 \Big/ \sum_\nu \Gamma^2_{(\nu)} \right\} \times \\ \times \left(G_{\alpha\rho} t^\alpha t^\rho - \lambda^2 \right) . \end{aligned} \tag{3.7}$$

Hence, we have the following theorem:

THEOREM 3.1. *In an even-dimensional special Kawaguchi space, the union curvature of a union curve is zero.*

THEOREM 3.2. *The union curvature vector* η^α *of a curve* C *relative to a congruence in an even-dimensional special Kawaguchi space is orthogonal to* C.

Proof. From the equation (3.1).

$$G_{\alpha\beta}\,\eta^\alpha\,\frac{du^\beta}{ds} = 0\,. \tag{3.8}$$

This proves the theorem.

THEOREM 3.3. *The necessary and sufficient condition that the geodesic curvature vector of a curve be equal to the union curvature vector relative to the congruence* λ^i *is that either the curve is asymptotic or the component of* λ^i *tangential to the subspace is tangential to the curve.*

Proof. The proof follows directly from the equation (3.1) and the fact that for an asymptotic curve

$$H^\nu_{\beta\gamma}\,\frac{du^\beta}{ds}\,\frac{du^\gamma}{ds} = 0 \qquad \text{(Yoshida [3])}\,. \tag{3.9}$$

4. GEODESIC CURVATURE OF A UNION CURVE

The angle θ, between the vector t^α and the unit tangent vector of the curve, is such that

$$\cos\,\theta \overset{\text{def}}{=} G_{\alpha\beta}\,\frac{du^\alpha}{ds}\,t^\beta\,(G_{\alpha\beta}\,t^\alpha\,t^\beta)^{-1/2} \tag{4.1}$$

since

$$G_{\alpha\beta}\,\frac{du^\alpha}{ds}\,\frac{du^\beta}{ds} = 1\,.$$

Corresponding to the given curve we define φ, the angle between the vectors t^α and p^β, by

$$\cos\varphi = G_{\alpha\beta}\,t^\alpha\,p^\beta\,(G_{\alpha\beta}\,t^\alpha\,t^\beta)^{-1/2}\,(G_{\alpha\beta}\,p^\alpha\,p^\beta)^{-1/2}\,, \tag{4.2}$$

whence using equation (3.4), we have

$$k_g\,(G_{\alpha\beta}\,t^\alpha\,t^\beta)^{-1/2}\cos\varphi = G_{\alpha\beta}\,t^\alpha\,p^\beta = t_\beta\,p^\beta. \tag{4.3}$$

The equations (3.7), (4.1), (4.3) and the relation $\lambda = t_\beta\,\dfrac{du^\beta}{ds}$ yield

$$k_u^2 = k_g^2 - 2\left\{\sum_\nu\left(H^\nu_{\beta\gamma}\,\frac{du^\beta}{ds}\,\frac{du^\gamma}{ds}\right)^2\Big/\sum_\nu\Gamma^2_{(\nu)}\right\}^{1/2}\times \tag{4.4}$$
$$\times\,k_g\,(G_{\alpha\rho}\,t^\alpha\,t^\rho)^{1/2}\cos\varphi + \left\{\sum_\nu\left(H^\nu_{\beta\gamma}\,\frac{du^\beta}{ds}\,\frac{du^\gamma}{ds}\right)^2\Big/\sum_\nu\Gamma^2_{(\nu)}\right\}\times$$
$$\times\,G_{\alpha\rho}\,t^\alpha\,t^\rho\,\sin^2\theta\,.$$

The following theorem is immediate from the above equation.

THEOREM 4.1. *A necessary and sufficient condition that the union curvatu e (of a curve of* Km*) relative to* λ^i *be expressed in the form*

$$k_u = k_g - \left\{ \sum_{\nu} \left(H^{\nu}_{\beta\gamma} \frac{du^{\beta}}{dr} \frac{du^{\gamma}}{ds} \right)^2 \Big/ \sum_{\nu} \Gamma^2_{(\nu)} \right\}^{1/2} (G_{\alpha\rho}\, t^{\alpha}\, t^{\rho})^{1/2} \sin\theta . \tag{4.5}$$

is that

$$\cos\varphi = \sin\theta. \tag{4.6}$$

When the condition (4.6) *is not satisfied the union curvature is given by* (4.4).

Since the union curvature of union curve is zero (Theorem 3.1), the above theorem reduces to the form:

THEOREM 4.2. *The geodesic curvature of the union curve relative to the congruence* λ^i *can be expressed as*

$$k_g = \left\{ \sum_{\nu} \left(H^{\nu}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} \right)^2 \Big/ \sum_{\nu} \Gamma^2_{(\nu)} \right\}^{1/2} (G_{\alpha\rho}\, t^{\alpha}\, t^{\rho})^{1/2} \sin\theta . \tag{4.7}$$

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