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On chain geometry of a projective line over a field of characteristic > 2 and quadratic order

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Geometria. — *On chain geometry of a projective line over a field of characteristic* > 2 *and quadratic order* (*). Nota (**) di VINCENZO DICUONZO, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — In questa Nota, mediante le cosiddette « catene » di certe rette proiettive finite, viene costruita una classe di spazi metrici finiti.

In this paper a class of finite metric spaces is constructed by the chains of certain finite projective lines.

Let r be a *projective line over a field* K *of characteristic* $p > 2$ *and order* $q = p^{2h}$ and denote by $\mathcal{C}$ an involutorial automorphism of K; in this way K contains a *subfield* K′ *of order* $q' = p^h$ consisting of the autoconjugate elements of K.

Introducing a nonhomogeneous coordinate system on r and denoting by $\bar{x}$ the conjugate of $x \in \mathrm{K}$, the following equations

$$a_{11}\, xx' + a_{12}\, x + a_{21}\, x' + a_{22} = 0 \tag{1}$$

and

$$a_{11}\, x\bar{x}' + a_{12}\, x + a_{21}\, \bar{x}' + a_{22} = 0\,, \tag{2}$$

with $a_{ij} \in \mathrm{K}$ and $a_{11}\, a_{22} - a_{12}\, a_{21} \neq 0$, represent a *projectivity* and an *antiprojectivity* on r, respectively; moreover we call *antiinvolution* an involutorial antiprojectivity, as an involutorial projectivity is called *involution.*

Now let us denote by r' the *subline* of r, defined over K′; it is said to be *chain* of r any subset of r, whose points are images of the ones of r' by a projectivity of r (see bibliography [1], p. 31 and [2], p. 94). Every chain contains $q' + 1$ points and there are $q'(q + 1)$ chains on r (see [1], pp. 110, 111).

We denote by Γ *the set of all chains of r.* For each element $\mathfrak{A}$ of Γ, there is an antiinvolution, whose fixed points are precisely the ones of $\mathfrak{A}$. A unique chain passes through any three distinct points of r (see [1], p. 111). It follows that, if $\mathfrak{A}$ and $\mathfrak{B}$ are two distinct chains, $|\mathfrak{A} \cap \mathfrak{B}| = 0, 1$ or 2. According to these possibilities, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are disjoint, tangent or intersecting, respectively. Two distinct chains are said to be orthogonal, if the corresponding antiinvolutions commute with one another.

We shall be concerned with some particularly important types of subsets of Γ later on: we introduce these now.

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A *hyperbolic pencil of* Γ is the set of all chains through two distinct points of r; a *parabolic pencil of* Γ is the set of all chains, mutually tangent, through a common point; an *elliptic pencil of* Γ is the set of all chains, such that the corresponding antiinvolutions transform two distinct points of r onto one another. A pencil of Γ contains $q'+1$, q', or $q'-1$ chains, according as it is hyperbolic, parabolic or elliptic (see [1], pp. 111, 112). The chains, orthogonal to the ones of a pencil $\mathfrak{F}$, form another pencil $\mathfrak{F}^*$, which is hyperbolic, parabolic or elliptic, according as $\mathfrak{F}$ is elliptic, parabolic or hyperbolic (see [1], p. 115): $\mathfrak{F}$ and $\mathfrak{F}^*$ are said to be orthogonal to one another. There are $\frac{1}{2}q(q+1)$ hyperbolic pencils of Γ, as many as the unordered pairs of points on r, and therefore $\frac{1}{2}q(q+1)$ elliptic pencils of Γ; moreover there are $(q+1)(q'+1)$ parabolic pencils, since there are $q+1$ points on r and for every point P $q'+1$ parabolic pencils relative to P.

At last we call *nonsingular bundle of* Γ the set of all chains, orthogonal to a same chain, and *singular bundle of* Γ the set of all chains through a point of r; there are $q'(q+1)$ nonsingular and $q+1$ singular bundles and each of them contains q or $q+q'$ chains respectively.

In order to construct a metric space Σ, we assume *the elements, the pencils and the bundles of* Γ as *planes, lines and points of the space* Σ, respectively. As a pencil of Γ, a *line* of Σ can be *hyperbolic, parabolic* or *elliptic*; as a bundle of Γ, a *point* of Σ is called *nonsingular* or *singular.*

A point P of Σ is said to belong to a line a or a plane α, if the bundle of Γ, corresponding to P, contains the pencil or the chain corresponding to a or α respectively; a line a of Σ belongs to a plane β, if the pencil corresponding to a contains the chain corresponding to β.

A *line a and* a *plane* β are called *incident* or *parallel*, according as a nonsingular or singular bundle of Γ contains the chain and the pencil corresponding to β and a respectively: likewise two lines of Σ are said to be incident or parallel.

As *orthogonality between two planes* of Σ we assume the *orthogonality between the corresponding chains.* It follows that the planes through a line a of Σ are orthogonal to the ones through another line a', which is called the *polar line* of a; a' is elliptic, parabolic or hyperbolic, according as a is hyperbolic, parabolic or elliptic. Moreover from the definition of nonsingular bundle of Γ it follows that the planes, orthogonal to the same plane α, pass through a nonsingular point A, which is called *the pole of* α; conversely, the planes through a nonsingular point A are orthogonal to a plane α, which is called *the polar plane of* A.

We must observe that *the incidence structure of* Σ *is weak*, because *there can be no plane through three noncollinear points* A, B, C. In fact this happens either when A, B, C are poles of planes represented by chains belonging to a singular bundle, or when A is singular and B, C are poles of planes through A. In order to obtain a space with the incidence structure of the projective three-space $S(3, q')$ over K′ we construct the so-called isotropic planes.

Let P be a *singular point* of Σ: we call *isotropic plane relative to* P, *the set* π *consisting of* P *and the poles of the planes through* P; as pole of π we assume the singular point P relative to it. By adding the isotropic planes, as it is easy to prove, the incidence structure of $S(3, q')$ is given to the space Σ.

Since two orthogonal chains of r are either disjoint or intersecting (see [1], p. 115), *two orthogonal planes of* Σ *intersect in a line, which is either elliptic or hyperbolic.*

If a and a' are two nonparabolic polar lines of Σ, α and β orthogonal planes through a, and γ and δ orthogonal planes through a', $\alpha, \beta, \gamma, \delta$ form an *autopolar tetrahedron.*

Since r with its chains is an egglike inversive plane $\mathfrak{M}$ (see [1], p. 112), because of a particular property of $\mathfrak{M}$ (see [3], n. 1), *with respect to the types of quadruplets of mutually orthogonal chains, the projective lines over fields of characteristic* > 2 *and quadratic order* q'^2 *may be divided into two classes, according as* $\frac{1}{2}(q'+1)$ *is odd or even.* Precisely, if $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ are four chains mutually orthogonal, three pencils of Γ, belonging to a nonsingular bundle and determined by unordered pairs of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, are hyperbolic or elliptic, according as $\frac{1}{2}(q'+1)$ is odd or even; correspondingly the other pencils are of different type and contain a common chain.

Now let us consider the antiinvolution $\mathfrak{I}(\mathfrak{A})$ relative to the chain $\mathfrak{A}$ representing a plane α. As is known, $\mathfrak{I}(\mathfrak{A})$ fixes $\mathfrak{A}$ pointwise, leaves the chains orthogonal to $\mathfrak{A}$ invariant and transforms chains, pencils and bundles of Γ onto chains, pencils and bundles of the same type respectively, preserving the orthogonality. This means that $\mathfrak{I}(\mathfrak{A})$ transforms planes, lines and points of Σ onto planes, lines and points of the same type respectively, leaving all points and lines of α fixed and all planes and lines orthogonal to α invariant.

We call $\mathfrak{I}(\mathfrak{A})$ *plane reflection of* Σ and assume *any product of plane reflections* as *motion of* Σ.

Since r with its chains is an egglike inversive plane over K', *each motion of* Σ *may be represented as product of at most five plane reflections* (see [3], n. 1).

Because of the last property of [3] n. 1, *the space* Σ *is the so-called metric-projective space of first type over* K': therefore we refer to [4] n. 1 for other properties of Σ.

In order to represent a plane Π *of* Σ, *independently, we now consider a bundle* $\mathfrak{R}$ *of* Γ *and assume the chains and the pencils of* $\mathfrak{R}$ *as lines and points of* Π *respectively.* In this way, *according as* $\mathfrak{R}$ *is singular or nonsingular,* Π *is Euclidean or hyperbolic.*

If $\mathfrak{R}$ is *singular and relative to a point* E *of* r, we denote the corresponding plane of Σ by Π_E. Since a pencil of $\mathfrak{R}(E)$ is either parabolic or hyperbolic, a *point of* Π_E is called either *singular* or *nonsingular* respectively.

A *line* a of Π_E is said to pass *through a point* P of Π_E, if the chain, corresponding to a, belongs to the pencil corresponding to P. *Two lines* a and b of Π_E are called *incident* or *parallel*, according as the point $a \cap b$ is nonsingular or singular.

Two lines of Π_E are said to be *orthogonal*, if the corresponding chains are orthogonal too. Because of the property that the chains of $\mathfrak{R}(E)$ orthogonal to a chain $\mathfrak{A}$ of $\mathfrak{R}(E)$ form a parabolic pencil, which is orthogonal to the parabolic pencil containing $\mathfrak{A}$, *the lines of* Π_E, *orthogonal to a line a, are parallel and each of them is orthogonal to the lines parallel to a.* It follows that *there are rectangles in* Π_E.

We now consider a line a of Π_E, the corresponding chain $\mathfrak{A}$ and the antiinvolution $\mathfrak{J}(\mathfrak{A})$ relative to $\mathfrak{A}$. As we saw for the plane reflections of Σ, $\mathfrak{J}(\mathfrak{A})$ leaves the line a pointwise fixed and the lines orthogonal to a invariant, and moreover transforms lines and points of Π_E onto lines and points of the same type respectively. Because of these properties we call $\mathfrak{J}(\mathfrak{A})$ *line reflection of* Π_E.

Any product of line reflections is called *motion of* Π_E.

As we know, r with its chains is the egglike inversive plane $\mathfrak{M}$ over K': this means that *the motion group* $G(E)$ *of* Π_E is isomorphic to the group, whose generators are the inversions of $\mathfrak{M}$ leaving a point of $\mathfrak{M}$ fixed, and therefore $G(E)$ *is isomorphic to the motion group of Euclidean plane over* K' (see [3], n. 2). For other properties of $G(E)$ we refer to [4], n. 1.

Lastly, in order to construct the hyperbolic plane Π_H over K' we consider a *nonsingular bundle* $\mathfrak{R}(\mathfrak{H})$ *of* Γ, that is the bundle of chains orthogonal to a same chain $\mathfrak{H}$.

As *lines and points of* Π_H we assume *the chains and the pencils of* $\mathfrak{R}(\mathfrak{H})$ respectively. Since, with respect to $\mathfrak{H}$, $\mathfrak{R}(\mathfrak{H})$ contains $\frac{1}{2}q'(q'+1)$ intersecting and $\frac{1}{2}q'(q'-1)$ disjoint chains, we call the *lines* of Π_H *hyperbolic* or *elliptic* correspondingly. Moreover a *point* of Π_H is called *hyperbolic, elliptic* or *singular*, according as the corresponding pencil of $\mathfrak{R}(\mathfrak{H})$ is hyperbolic, elliptic or parabolic.

A line of Π_H is said to pass through a point P, if the corresponding chain belongs to the pencil representing P.

Two distinct *lines* of Π_H, passing through a singular point, are called *parallel.* Two *lines* of Π_H are said to be *orthogonal*, if the corresponding chains are orthogonal too.

The chains of $\mathfrak{R}(\mathfrak{H})$ orthogonal to a given chain $\mathfrak{A}$ of $\mathfrak{R}(\mathfrak{H})$ form a nonparabolic pencil; conversely the chains of a nonparabolic pencil of $\mathfrak{R}(\mathfrak{H})$ are orthogonal to a chain of $\mathfrak{R}(\mathfrak{H})$. This means that *the lines, orthogonal to a line a, pass through a nonsingular point* A, *which is called the pole of a*; conversely *the lines through a nonsingular point* A *are orthogonal to a line a which is called the polar line of* A.

We must observe that *the incidence structure of* Π_H *is weak*, because *there can be no line passing through two distinct points* A *and* B: in fact this happens when A and B are two nonsingular points, poles of two parallel lines a and b, or when A is a singular point and B is the pole of a line through A.

In order to avoid this fact we introduce some new lines, which will be called isotropic lines.

Let P be a *singular point* of Π_H: we define *isotropic line relative to* P, *the set consisting of* P *and the poles of the (hyperbolic) lines through* P.

We saw that there are two types of quadruplets of mutually orthogonal chains of r. Correspondingly *the triples of mutually orthogonal chains of* $\mathfrak{R}(\mathfrak{H})$ *may be of four types and there are two of these, if* $\frac{1}{2}(q'+1)$ *is odd, while the other two if* $\frac{1}{2}(q'+1)$ *is even*: precisely, if $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are three mutually orthogonal chains of $\mathfrak{R}(\mathfrak{H})$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ determine three pencils $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ and the number of the hyperbolic ones is odd or even, according as $\frac{1}{2}(q'+1)$ is odd or even. This means that, *with respect to the three-right-angled triangles, the planes* Π_H *may be divided into two classes, according as* $\frac{1}{2}(q'+1)$ *is odd or even*: precisely, the number of the hyperbolic sides of an above triangle is odd or even, according as $\frac{1}{2}(q'+1)$ is odd or even.

We now consider a line a of Π_H, the corresponding chain $\mathfrak{A}$ and the antiinvolution $\mathfrak{J}(\mathfrak{A})$ relative to $\mathfrak{A}$. As for Π_E, we call $\mathfrak{J}(\mathfrak{A})$ *line reflection of* Π_H and define *motion of* Π_H *any product of line reflections*.

The motions of Π_H *form a group* G_H isomorphic to the group $\bar{G}$, whose generators are the inversions of a nonsingular bundle of the egglike inversive plane $\mathfrak{M}$ over K′ and therefore (see [3], n. 3) G_H is *isomorphic to the motion group of the hyperbolic plane over* K′; it follows that *any motion of* Π_H *may be represented as product of two line reflections*.

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