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Growth Properties of Functional Differential Equations

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Equazioni differenziali funzionali. — *Growth Properties of Functional Differential Equations.* Nota di G. S. LADDE, presentata (*) del Socio G. SANSONE.

RIASSUNTO. — Impiegando le nozioni di soluzioni globalmente uniformemente stabili e di soluzioni uniformemente lentamente crescenti in variazione sono costruite delle funzioni di Lyapunov.

Queste funzioni sono impiegate per lo studio della cresenza delle soluzioni di equazioni differenziali funzionali.

1. INTRODUCTION

In this paper, we concern ourselves with the study of certain growth properties of solutions of functional differential equations. For this purpose, we use the notions of global uniform stability in variation and global uniform slowly-growing in variation. The main techniques are similar to that of the previous work in [2]. All our results generalize and extend the results of Brauer and Strauss [1].

2. NOTATIONS AND DEFINITIONS

Let R^+ denote the interval $[0, \infty]$, let R^n denote the n-dimensional real Euclidean space and for $\tau \geq 0$, let $\mathcal{C}^n = C[[-\tau, 0], R^n]$ denote the space of continuous functions with domain $[-\tau, 0]$ and range in R^n. For any $\varphi \in \mathcal{C}^n$, define the norm

$$\|\varphi\|_0 = \sup_{-\tau \leq s \leq 0} \|\varphi(s)\|,$$

where $\|-\|$ is any convenient norm in R^n. Suppose that $x \in C[[-\tau, \infty), R^n]$. For any $t \geq 0$, we shall let x_t denote the element of $\mathcal{C}^n$ defined by

$$x_t(s) = x(t+s), \qquad -\tau \leq s \leq 0.$$

With this notation, we write the following differential systems

$$x' = f(t, x) \quad , \quad x(t_0) = x_0, \tag{2.1}$$

and

$$y' = F(t, y_t) \quad , \quad y_{t_0} = \varphi_0, \tag{2.2}$$

where $f \in C[R^+ \times R^n, R^n]$, $F \in C[R^+ \times \mathcal{C}^n, R^n]$. Furthermore, assume that $\partial f/\partial x$ exists and is continuous for $(t, x) \in R^+ \times R^n$.

(*) Nella seduta del 26 novembre 1973.

Notice that the functional differential system (2.2) may be considered as a perturbation of (2.1) since it may be written in the form

$$(2.3) \qquad y' = f(t, y) + R(t, y, y_t) \quad , \quad y_{t_0} = \varphi_0 ,$$

where $R(t, y, y_t) = F(t, y_t) - f(t, y)$. We are going to use this remark in our considerations subsequently.

Let $x(t, t_0, x_0)$ and $y(t_0, \varphi_0)(t)$ denote any solutions of (2.1) and (2.2) respectively. Let $\Phi(t, t_0, x_0)$ denote the fundamental matrix solution of the variational system

$$(2.4) \qquad z' = f_x(t, x(t, t_0, x_0))\, z$$

with respect to the solution $x(t, t_0, x_0)$ of (2.1) such that

$$\Phi(t_0, t_0, x_0) = I \quad \text{(Identity matrix).}$$

We recall that

$$\Phi(t, t_0, x_0) = \frac{\partial x}{\partial x_0}(t, t_0, x_0)$$

and

$$-\Phi(t, t_0, x_0) \cdot f(t_0, x_0) = \frac{\partial x}{\partial t_0}(t, t_0, x_0).$$

We define the concepts of slowly-growing and global uniform stability in variation.

DEFINITION 2.1. *The type number* of a vector-valued function $z(t)$ is defined by

$$m = \limsup_{t \to \infty} \frac{\log \|z(t)\|}{t}.$$

DEFINITION 2.2. If the m in the Definition 2.1 is non-negative, then the function $z(t)$ is said to be *slowly-growing*.

DEFINITION 2.3. The solution $x \equiv 0$ of (2.1) is said to be *globally uniformly slowly-growing in variation*, if for every $\varepsilon > 0$, there exists a constant $K(\varepsilon) = K > 0$, such that

$$\|\Phi(t, t_0, x_0)\| \leq K \exp[\varepsilon(t - t_0)], \qquad \text{for} \quad t \geq t_0$$

whenever

$$\|x_0\| < \infty.$$

DEFINITION 2.4. The solution $x \equiv 0$ of (2.1) is said to be *globally uniformly stable in variation* if there exists M such that

$$\|\Phi(t, t_0, x_0)\| \leq M, \qquad \text{for} \quad t \geq t_0$$

whenever

$$\|x_0\| < \infty.$$

3. Construction of Lyapunov Functions

In this section, we construct some Lyapunov functions which will be used afterwards to study growth properties of (2.2).

Theorem 3.1. *Assume that the solution* $x \equiv 0$ *of* (2.1) *is globally uniformly slowly-growing in variation. Then there exists a function* $V_1(t, x)$ *having the following properties*:

(i) $V_1(t, x)$ *is defined and continuous on* $R^+ \times R^n$;

(ii) *for* $(t, x) \in R^+ \times R^n$,

$$\|x\| \leq V_1(t, x) \leq K \|x\|;$$

(iii) *for* $(t, x), (t, y) \in R^+ \times R^n$,

$$|V_1(t, x) - V_1(t, y)| \leq K \|x - y\|;$$

(iv) *for* $(t, x) \in R^+ \times R^n$.

$$D^+ V_1(t, x) \leq \varepsilon V(t, x).$$
(2.1)

Proof. The globally uniformly slowly growing in variation of the trivial solution of (2.1), together with the Theorem 2.6.4 [3, pp. 78] implies that

$$\|x(t, t_0, x_0)\| \leq K \|x_0\| \exp[\varepsilon(t - t_0)], \qquad t \geq t_0, \tag{3.1}$$

whenever

$$\|x_0\| < \infty.$$

This shows that the trivial solution of (2.1) is globally uniformly slowly growing.

Define

$$V_1(t, x) = \sup_{\sigma \geq 0} \|x(t + \sigma, t, x)\| e^{-\varepsilon\sigma}, \qquad \text{for} \quad (t, x) \in R^+ \times R^n. \tag{3.2}$$

where $x(t + \sigma, t, x)$ is the solution of (2.1) through $(t, x) \in R^+ \times R^n$.

The verification of stated properties of $V_1(t, x)$ in the theorem can be given in similar lines as in the Theorem 3.6.1 [3, pp. 158], so we omit the details.

We state another converse theorem whose proof follows immediately from the Theorem 3.1 [2].

Theorem 3.2. *Assume that the trivial solution of* (2.1) *is globally uniformly stable in variation. Then there exists a function* $V_2(t, x)$ *possessing the following properties*:

(i) $V_2(t, x)$ *is defined and continuous on* $R^+ \times R^n$;

(ii) *for* $(t, x) \in R^+ \times R^n$,

$$\|x\| \leq V_2(t, x) \leq M \|x\|;$$

(iii) *for* $(t, x), (t, y) \in R^+ \times R^n$,

$$|V_2(t, x) - V_2(t, y)| \leq M \|x - y\|;$$

(iv) $D^+ V_2(t, x) \leq 0$, for $(t, x) \in R^+ \times R^n$.
(2.1)

4. Main Results

In order to study growth properties of (2.2), we need to extend $V_i(t, x)$, for $i = 1, 2$ as follows:

Define

$$V(t, x) = \begin{cases} V_i(t, x), & \text{for } (t, x) \in R^+ \times R^n, \\ V_i(0, x), & \text{for } (t, x) \in [-\tau, 0] \times R^n. \end{cases} \tag{4.1}$$

It is easy to verify that,

(I) $V \in C[[-\tau, \infty) \times R^n, R^+]$ and V is Lipschitzian in x with the same Lipschitz constant as that of V_i for $i = 1,2$.

(II) $V(t, x)$ is positive definite and decreasing in x for $(t, x) \in [-\tau, \infty) \times R^n$, with the same functions a and $b \in K$ as that of V_i for $i = 1,2$.

THEOREM 4.1. *Let the solution* $x \equiv 0$ *of* (2.1) *be globally uniformly slowly-growing in variation. Suppose that* $R(t, \varphi(0), \varphi)$ *in* (2.3) *satisfies*

$$\|R(t, \varphi(0), \varphi)\| \leq G(t, \|\varphi(0)\|, \|\varphi\|), \tag{4.2}$$

for $(t, \varphi) \in R^+ \times \mathcal{C}^n$, *where* $G \in C[R^+ \times R^+ \times \mathcal{C}_+, R^+]$ *and* $G(t, \sigma(0), \sigma)$ *is increasing in* $\sigma(0)$ *and* σ *for* $\sigma \in \mathcal{C}_+$. *Furthermore, we assume that the maximal solutions of the functional differential equation*

$$u' = \varepsilon u + KG(t, u, u_t) \quad , \quad u_{t_0} = \sigma_0 \tag{4.3}$$

is globally uniformly slowly-growing, where K *is the same constant as that of Theorem* 3.1. *Then, all solutions of* (2.2) *are globally uniformly slowly-growing.*

Proof. By Theorem 3.1. there exists a function $V(t, x)$ having the properties (I) and (II) stated in the above paragraph. Using (4.1), (4.2) and increasing character of G and following the proof of the Theorem 4.1 [2], it is easy to get the following inequality

$$D^+ V(t, \varphi(0), \varphi) \leq \varepsilon V(t, \varphi(0)) + KG(t, V(t, \varphi(0)), V_t), \tag{4.4}$$
(2.2)

for $(t, \varphi) \in R^+ \times \mathcal{C}^n$.

Let $y(t_0, \varphi_0)(t)$ be any solution of (2.2), such that

$$V(t_0 + s, \varphi_0(s)) \leq K \|\varphi_0(s)\|, \quad \text{for } s \in [-\tau, 0] \tag{4.5}$$

Set $\varphi = y_t(t_0, \varphi_0)$ and $\varphi(0) = y(t_0, \varphi_0)(t)$. Suppose that $x(t+\sigma, t, \varphi(0))$ is any solution of (2.1), through $(t, \varphi(0))$ for $\sigma \geq 0$.

Let $r(t_0, \sigma_0)$ be the maximal solution of the functional differential equation (4.3) with

$$\sigma_0(s) = K\,\|\varphi_0(s)\|, \qquad \text{for} \quad s \in [-\tau, 0]. \tag{4.6}$$

This together with (4.4), (4.5), (4.3) and Theorem 8.1.4 [3, pp. 85–86], we have

$$V(t, y(t_0, \varphi_0))(t) \leq r(t_0, \sigma_0)(t), \qquad t \geq t_0. \tag{4.7}$$

From the property (II) and (4.7), we get

$$\|y(t_0, \varphi_0)(t)\| \leq r(t_0, \sigma_0)(t), \qquad t \geq, t_0, \tag{4.8}$$

noting the fact that $b(r) = r$, $a(r) = Kr$.

Hence from the hypothesis that $r(t_0, \sigma_0)(t)$ is globally uniformly slowly growing, we have the desired result.

COROLLARY 4.1. *Let the solution $x = 0$ of (2.1) be globally uniformly slowly growing in variation. Suppose that $R(t, \varphi(0), \varphi)$ satisfies*

$$\|R(t, \varphi(0), \varphi)\| \leq \lambda(t) \int_{-\tau}^{0} \|\varphi(s)\|\, ds + \mu(t), \tag{4.9}$$

for $(t, \varphi) \in R^+ \times \mathcal{C}^n$, where μ is slowly growing and

$$\Lambda(t) = (t - t_0)^{-1} \int_{t_0}^{t} \lambda(s)\, ds \to 0 \qquad \text{as} \quad t \to \infty. \tag{4.10}$$

Then the conclusion of the previous Theorem remains true.

Proof. Following the proof of the previous Theorem, the functional differential equation (4.3) reduces to

$$u' = \varepsilon u + \lambda(t)\, K\tau \sup_{-\tau \leq s \leq 0} u(t+s) + K\mu(t), \tag{4.11}$$

with

$$u_{t_0} = \sigma_0 \qquad \text{for any} \quad \sigma_0 \in \mathcal{C}_+.$$

Let $y(t_0, \varphi_0)(t)$ be any solution of (2.2) such that

$$V(t_0 + s, \varphi_0(s)) \leq K\,\|\varphi_0(s)\|, \qquad \text{for} \quad s \in [-\tau, 0]. \tag{4.12}$$

The maximal solution of

$$u' = g(t, u) = [\varepsilon + K\tau\lambda(t)]\, u + K\mu(t)$$

with

$$u(t_0) = \|\sigma_0\|_0, \qquad \text{where} \quad \sigma_0(s) = \mathrm{K}\,\|\varphi_0(s)\|, \qquad s \in [-\tau, 0],$$

exists for $t \geq t_0$. Applying Lemma 2 [2], and hypothesis on μ, we get

$$(4.13) \qquad r(t_0, \sigma_0)(t) \leq [u_0 + \mathrm{K}^2(t - t_0)] \exp[\varepsilon(t - t_0) + \mathrm{K}\tau\Lambda(t)(t - t_0)],$$

for $t \geq t_0$, where $r(t_0, \sigma_0)(t)$ is any solution of (4.11) with

$$\sigma_0(s) = \mathrm{K}\,\|\varphi_0(s)\|, \qquad \text{for} \quad s \in [-\tau, 0].$$

Our aim is to show that the maximal solution of (4.11) is slowly-growing, so that the conclusion of the corollary follows immediately by the previous Theorem. To prove this, by the hypothesis (4.10), we can choose T such that

$$\mathrm{K}\tau\Lambda(t) < \varepsilon, \qquad \text{for} \quad t \geq \mathrm{T},$$

and choose $\mathrm{K}_1 = \mathrm{K}_1(\varepsilon, \|\sigma_0\|_0)$ so that

$$\|\sigma_0\|_0 + \mathrm{K}^2(t - t_0) \leq \mathrm{K}_1 \exp[\varepsilon(t - t_0)],$$

for $t \geq \mathrm{T}$. This together with (4.13), yields

$$r(t_0, \sigma_0)(t) \leq \mathrm{K}_1 \exp[3\,\varepsilon(t - t_0)]$$

for $t \geq \mathrm{T}$, which implies that the maximal solution of (4.11) is globally uniformly slowly-growing. Hence the corollary is proved.

Remark 4.1. In the hypothesis of the corollary, if (4.10) is replaced by

$$\Lambda(t) \to \mathrm{C} \qquad \text{as} \quad t \to \infty,$$

for some finite $\mathrm{C} > 0$. Assuming all hypotheses of the corollary being the same, the conclusion of the corollary holds, provided that the time lag satisfies the following condition

$$0 < \tau < \frac{\varepsilon}{2\,\mathrm{KC}}, \qquad \text{for every} \quad \varepsilon > 0.$$

This remark gives the estimate for time lag.

Remark 4.2. Assume all hypotheses of the corollary hold except that (4.9) is replaced by

$$\|\mathrm{R}(t, \varphi(0), \varphi)\| \leq \lambda(t) \sup_{-\tau \leq s \leq 0} \|\varphi(s)\| + \mu(t),$$

for $(t, \varphi) \in \mathrm{R}^+ \times \mathscr{C}^n$. Then, the conclusion of the corollary remains true.

Note that the Remark 4.2 is the natural generalization of the Theorem 4 [1] for functional differential equations.

Now, we obtain some estimates for the rate of growth of unbounded solutions of (2.2). For this we assume global uniform stability in variation.

THEOREM 4.2. *Let the trivial solution of* (2.1) *be globally uniformly stable in variation. Suppose that* $R(t, \varphi(0), \varphi)$ *in* (2.3) *satisfies*

$$\|R(t, \varphi(0), \varphi)\| \leq \lambda_1(t)\|\varphi(0)\| + \lambda_2(t)\int_{-\tau}^{0}\|\varphi(s)\|\,ds, \tag{4.14}$$

for $(t, \varphi) \in R^+ \times \mathcal{C}^n$, *where* $\lambda_i \in C[R^+, R^+]$, $i = 1, 2$ *and satisfy an inequality of the form*

$$[\lambda_1(t) + \tau\lambda_2(t)] \leq L\,\frac{P'(t)}{P(t)}, \tag{4.15}$$

for large $t \geq t_0$ *and* $P(t)$ *is some polynomial of degree* $n \geq 1$. *Then, the solutions of* (2.2) *do not grow more rapidly than a polynomial of degree* mn, *as* $t \to \infty$, *where* $ML \leq m$, *for some positive integer* m.

Proof. By Theorem 3.2, there exists a function $V(t, x)$ having the properties (I) and (II) as stated before. Now following the proof of the Theorem 4.1, it is easy to arrive at the inequality, after certain simple computations,

$$D^+_{(2.2)} V(t, \varphi(0), \varphi) \leq M[\lambda_1(t)V(t, \varphi(0)) + \tau\lambda_2(t)\sup_{-\tau\leq s\leq 0} V(t+s, \varphi(s))] \tag{4.16}$$

for $(t, \varphi) \in R^+ \times \mathcal{C}^n$.

Let $y(t_0, \varphi_0)(t)$ be any solution of (2.2) such that

$$V(t_0 + s, \varphi_0(s)) \leq M\|\varphi_0(s)\|, \qquad \text{for} \quad s \in [-\tau, 0], \tag{4.17}$$

for $\varphi_0 \in \mathcal{C}^n$.

Let $r(t_0, \sigma_0)(t)$ be any solution of

$$u' = M[\lambda_1(t)u + \tau\lambda_2(t)\sup_{-\tau\leq s\leq 0} u(t+s)] \tag{4.18}$$

with

$$\sigma_0(s) = M\|\varphi_0(s)\|, \qquad \text{for} \quad \varphi_0 \in \mathcal{C}^n.$$

The maximal solution of

$$u' = g(t, u) = M[\lambda_1(t) + \tau\lambda_2(t)]u, \qquad u(t_0) = \|\sigma_0\|_0,$$

is

$$u(t, t_0, u_0) = u_0 \exp\left[M\int_{t_0}^{t}(\lambda_1(s) + \tau\lambda_2(s))\,ds\right].$$

Hence by the application of Lemma 2 [2], we have,

$$r(t_0, \sigma_0)(t) \leq \|\sigma_0\|_0 \exp\left[M\int_{t_0}^{t}(\lambda_1(s) + \tau\lambda_2(s))\,ds\right].$$

This together with (4.15), yields

$$(4.19) \qquad r(t_0, \sigma_0)(t) \leq \|\sigma_0\|_0 \exp \mathrm{ML}\, ln \left[\frac{\mathrm{P}(t)}{\mathrm{P}(t_0)}\right] = \|\sigma_0\|_0 \left[\frac{1}{\mathrm{P}(t_0)}\right]^{\mathrm{ML}} [\mathrm{P}(t)]^{\mathrm{ML}}$$

for large $t \geq t_0$. Then as in Theorem 4.1, with necessary modifications, we obtain the inequality

$$\|y(t_0, \varphi_0)(t)\| \leq \|\sigma_0\|_0 [\mathrm{P}(t_0)]^{-\mathrm{ML}} [\mathrm{P}(t)]^m$$

which guarantees the stated growth property and hence the proof is complete.

Remark 4.3. In the Theorem 4.2, if the condition (4.14) is replaced by the condition

$$\|\mathrm{R}(t, \varphi(0), \varphi)\| \leq \lambda_2(t) \sup_{-\tau \leq s \leq 0} \|\varphi(s)\|.$$

Assuming all hypotheses of the Theorem being the same, then the conclusion of the Theorem remains true.

Observe that this remark is a natural generalization of the Theorem 3 in [1] for functional differential equation

THEOREM 4.3. *Let the trivial solution of* (2.1) *be globally uniformly stable in variation. Suppose that* $\mathrm{R}(t, \varphi(0), \varphi)$ *satisfies the relation*

$$\|\mathrm{R}(t, \varphi(0), \varphi)\| \leq \lambda(t), \qquad t \geq 0 \quad , \quad \|\varphi\|_0 < \infty.$$

Let

$$\Lambda(t) = [g(t)]^{-1} \int_{t_0}^{t} \lambda(s)\, ds;$$

(i) *if* $\Lambda(t)$ *is bounded then any solution of* (2.2) *satisfies*

$$y(t_0, \varphi_0)(t) \to \mathrm{O}(g(t)) \qquad \text{as} \quad t \to \infty;$$

(ii) *if* $\Lambda(t) \to 0$ *as* $t \to \infty$, *then each solution of* (2.2) *satisfies*

$$y(t_0, \varphi_0)(t) \to o(g(t)) \qquad \text{as} \quad t \to \infty.$$

Proof. Proof of this Theorem is analogous to the previous Theorem and nence we omit the details.

Remark 4.4. If all the hypotheses of the Theorem 4.3 hold and take $g(t) = t$, then our result generalizes the Theorem 2 in [1], for functional differential equations.

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