
ATTI ACCADEMIA NAZIONALE DEI LINCEI
CLASSE SCIENZE FISICHE MATEMATICHE NATURALI
RENDICONTI

RODNEY S. RAMBALLY

**Boundedness of solutions of perturbed systems of
differential equations**

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche,
Matematiche e Naturali. Rendiconti, Serie 8, Vol. 55 (1973), n.5, p. 348–355.*

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_55_5_348_0>

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Equazioni differenziali ordinarie. — *Boundedness of solutions of perturbed systems of differential equations.* Nota di RODNEY S. RAMBALLY, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Si considerano generali sistemi di equazioni differenziali ordinarie ed in particolare lineari e si danno condizioni sufficienti alle quali devono soddisfare perturbazioni dei dati, determinanti la soluzione dei detti sistemi, affinché questa si mantenga entro confini prescritti.

NOTATION

α	Alpha	$\{ \}$	set notation
β	Beta	$/$	derivative with respect to time (d/dt)
γ	Gamma	$ $	vector norm
δ	Delta	$\mathbb{R}$	set of real numbers
ε	Epsilon	$>$	greater than
τ	Tau	$\geq$	greater than or equal to
ω	Omega	$<$	less than
∞	Infinity	$\leq$	less than or equal to
ϵ	is an element of	a/b	a divided by b .

1. We consider the systems of differential equations

$$(1) \quad x' = f(t, x)$$

$$(2) \quad y' = f(t, y) + g(t, y)$$

where x, y, f and g are n -dimensional vectors and where $f(t, 0) = 0$, thereby making $x = 0$ a solution of (1). Let $|\cdot|$ denote a convenient vector norm and a corresponding matrix norm.

Assume there exists a solution $x(t)$ of (1) which is defined for all $t \geq t_0$. Recall that the solution $x = 0$ of (1) is said to be

a) *Uniformly stable* if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x(t)$ satisfies $|x(t_1)| < \delta$ for some $t_1 \geq t_0$, then $|x(t)| < \varepsilon$ for all $t \geq t_1$;

b) *uniformly asymptotically stable* if, in addition to (a), there is a $\delta > 0$ and for each $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that whenever $|x(t_1)| < \delta$ for some $t_1 \geq t_0$, then $|x(t)| < \varepsilon$ for all $t \geq t_1 + T$.

We shall usually denote the solution of (1) through (t_0, x_0) by $x(t, t_0, x_0)$ and similarly the solution of (2) through (t_0, y_0) by $y(t, t_0, y_0)$.

(*) Nella seduta del 26 novembre 1973.

2. For the first theorem we assume that f satisfies a uniform Lipschitz condition on $D_M = \{(t, x) : t \geq 0, |x| \leq M, M > 0\}$. Let L be the Lipschitz constant.

THEOREM 1. *Let $0 \leq t < \infty$ and $0 < r < H$. Assume there exists $\alpha > 0$ such that whenever $|y| \leq \alpha$, $|g(t, y)| \leq \omega(t, |y|)$, where $\omega(t, r)$ is a continuous, scalar function which is monotone, nondecreasing in r for each fixed t and which satisfies*

$$\int_a^\infty \omega(t, c) dt < \infty \quad \text{for } 0 < c < H, \quad 0 \leq a < \infty.$$

Assume that the solution $x = 0$ of (1) is uniformly asymptotically stable. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ and $T > 0$ such that if $t_0 \geq T$ and $|y_0| < \delta$, then $|y(t, t_0, y_0)| < \varepsilon$ for all $t \geq t_0$.

We need the following lemma in order to prove the theorem.

LEMMA 1. *Let $x(t, \bar{t}_0, \bar{y}_0)$ and $y(t, \bar{t}_0, \bar{y}_0)$ be solutions of (1) and (2) respectively. Assume that $|y(t, \bar{t}_0, \bar{y}_0)| \leq \alpha$ for $t \in [\bar{t}_0, \bar{t}_0 + \tau]$ for some $\tau > 0$. Then*

$$|y(t, \bar{t}_0, \bar{y}_0) - x(t, \bar{t}_0, \bar{y}_0)| \leq e^{L\tau} \int_{\bar{t}_0}^t \omega(s, \alpha) ds$$

for $t \in [\bar{t}_0, \bar{t}_0 + \tau]$.

Proof. For simplicity, we denote $y(s, \bar{t}_0, \bar{y}_0)$ and $x(s, \bar{t}_0, \bar{y}_0)$ by $y(s)$ and $x(s)$ respectively. Now

$$\begin{aligned} & |y(t, \bar{t}_0, \bar{y}_0) - x(t, \bar{t}_0, \bar{y}_0)| \\ &= \left| \bar{y}_0 + \int_{\bar{t}_0}^t [f(s, y(s)) + g(s, y(s))] ds \right. \\ & \quad \left. - \bar{y}_0 - \int_{\bar{t}_0}^t f(s, x(s)) ds \right| \\ &= \left| \int_{\bar{t}_0}^t [f(s, y(s)) - f(s, x(s)) + g(s, y(s))] ds \right| \\ &\leq \int_{\bar{t}_0}^t L |y(s) - x(s)| ds + \int_{\bar{t}_0}^t \omega(s, |y(s)|) ds. \end{aligned}$$

Hence (Brauer [1], Lemma 2)

$$\begin{aligned} & |y(t, t_0, y_0) - x(t, t_0, y_0)| \\ & \leq \int_{t_0}^t \omega(s, |y(s)|) e^{L(t-s)} ds \\ & \leq e^{L\tau} \int_{t_0}^t \omega(s, \alpha) ds. \end{aligned}$$

Proof of Theorem 1. Let all constants corresponding to (1) be starred. Let $0 < \varepsilon \leq \alpha$. Choose $\delta = \delta(\varepsilon) = \delta^*(\varepsilon/2)$ (where δ^* is as in the definition of uniform stability of the zero solution of (1)) such that $\delta < \varepsilon$. Let $\tau = \tau(\varepsilon) = T^*(\delta/2)$ and choose T large enough such that

$$\int_T^\infty \omega(s, \alpha) ds < \frac{\delta}{2e^{L\tau}}.$$

Finally choose $t_0 \geq T$ and $|y_0| < \delta$. Note that for such a y_0 , $|x(t, t_0, y_0)| < \varepsilon/2$ for $t \geq t_0$ and $|x(t, t_0, y_0)| < \delta/2$ for $t \geq t_0 + T^*(\delta/2)$. Provided that $|y(t, t_0, y_0)| < \varepsilon \leq \alpha$ for $t \in [t_0, t_0 + \tau]$, we have, using Lemma 1,

$$\begin{aligned} |y(t, t_0, y_0)| & \leq |y(t, t_0, y_0) - x(t, t_0, y_0)| + |x(t, t_0, y_0)| \\ & \leq e^{L\tau} \int_{t_0}^t \omega(s, \alpha) ds + \frac{\varepsilon}{2} \\ & \leq e^{L\tau} \int_T^t \omega(s, \alpha) ds + \frac{\varepsilon}{2} \\ & < \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus $y(t, t_0, y_0)$ can be continued to the whole of $[t_0, t_0 + \tau]$ and on this interval $|y(t, t_0, y_0)| < \varepsilon$. Let $y_1 = y(t_0 + \tau, t_0, y_0)$. Then

$$\begin{aligned} |y_1| & \leq |y_1 - x(t_0 + \tau, t_0, y_0)| + |x(t_0 + \tau, t_0, y_0)| \\ & < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Assume that for some $m > 0$, $|y(t, t_0, y_0)| < \varepsilon$ for $t \in [t_0, t_0 + m\tau]$ and that $|y(t_0 + m\tau, t_0, y_0)| < \delta$. Let $y_m = y(t_0 + m\tau, t_0, y_0)$. We would like to show that $|y(t, t_0, y_0)| < \varepsilon$ for $t \in [t_0 + m\tau, t_0 + (m+1)\tau]$. Let us consider $y(t, t_0 + m\tau, y_m)$ on $[t_0 + m\tau, t_0 + (m+1)\tau]$. Provided that

$|y(t, t_0 + m\tau, y_m)| < \varepsilon$ on $[t_0 + m\tau, t_0 + (m+1)\tau]$ we have

$$\begin{aligned} |y(t, t_0 + m\tau, y_m)| &\leq |y(t, t_0 + m\tau, y_m) - x(t, t_0 + m\tau, y_m)| \\ &\quad + |x(t, t_0 + m\tau, y_m)| \\ &\leq e^{L\tau} \int_{t_0 + m\tau}^t \omega(s, \alpha) ds + \frac{\varepsilon}{2} \\ &\leq e^{L\tau} \int_{\bar{t}}^t \omega(s, \alpha) ds + \frac{\varepsilon}{2} \\ &< \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus $|y(t, t_0, y_0)| < \varepsilon$ on the entire interval $[t_0 + m\tau, t_0 + (m+1)\tau]$. Finally, let $y_{m+1} = y(t_0 + (m+1)\tau, t_0, y_0)$. Then

$$\begin{aligned} |y_{m+1}| &\leq |y(t_0 + (m+1)\tau, t_0 + m\tau, y_m) \\ &\quad - x(t_0 + (m+1)\tau, t_0 + m\tau, y_m)| \\ &\quad + |x(t_0 + (m+1)\tau, t_0 + m\tau, y_m)|. \end{aligned}$$

This follows because $y(t, t_0, y_0)$ can be continued to $t_0 + (m+1)\tau$, thereby making $y(t_0 + (m+1)\tau, t_0, y_0)$ and $y(t_0 + (m+1)\tau, t_0 + m\tau, y_m)$ equal. Thus

$$|y_{m+1}| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Since $|y(t, t_0, y_0)| < \varepsilon$ on every interval $[t_0 + m\tau, t_0 + (m+1)\tau]$, then $|y(t, t_0, y_0)| < \varepsilon$ on $[t_0, \infty)$. The proof of the theorem is now complete.

We shall now consider the case when our homogeneous system (1) is linear. In Theorem 2, we study the systems

$$(3) \quad x' = A(t)x$$

$$(4) \quad y' = A(t)y + f(t, y) + b(t),$$

where $A(t)$ and f are continuous. Let $X(t)$ be a fundamental matrix of (3) and let X_1 be the subspace of points in R^n which are values for $t = 0$ of the bounded solutions of (3). Let X_2 be a fixed subspace complementary to X_1 and let P_1 and P_2 be the corresponding projections of R^n onto X_1 and X_2 .

THEOREM 2. Assume that

$$\int_0^t |X(t)P_1X^{-1}(s)|^q ds + \int_t^\infty |X(t)P_2X^{-1}(s)|^q ds \leq K$$

for some positive constant K . Let $f(t, y)$ be a continuous function such that

$$|f(t, y_1) - f(t, y_2)| \leq \gamma |y_1 - y_2| \quad \text{where } \gamma K < 1,$$

$0 \leq t < \infty$, $|y_1| \leq \beta$, $|y_2| \leq \beta$ for some constant β . Assume that $|b(t)| \in L^p[0, \infty)$, where $1 < p < \infty$ and $1/p + 1/q = 1$ and let the L^p norm of $|b(t)|$, $|b|_p$, be such that

$$|b|_p \leq \frac{\beta(1 - \gamma K)}{2 K^{1/q}}.$$

Then there exists a unique solution $y(t)$ of (4) such that $|y(t)| \leq \beta$.

Proof. Let $y(t)$ be any continuous function such that $|y(t)| \leq \beta$ and let

$$\begin{aligned} \tau y(t) &= \int_0^t X(t) P_1 X^{-1}(s) (b(s) + f(s, y(s))) ds \\ &\quad - \int_t^\infty X(t) P_2 X^{-1}(s) (b(s) + f(s, y(s))) ds. \end{aligned}$$

We claim that τ is a contraction mapping. This follows because

$$\begin{aligned} |\tau y_1 - \tau y_2| &= \left| \int_0^t X(t) P_1 X^{-1}(s) (b(s) + f(s, y_1(s))) ds \right. \\ &\quad \left. - \int_t^\infty X(t) P_2 X^{-1}(s) (b(s) + f(s, y_1(s))) ds \right. \\ &\quad \left. - \int_0^t X(t) P_1 X^{-1}(s) (b(s) + f(s, y_2(s))) ds \right. \\ &\quad \left. + \int_t^\infty X(t) P_2 X^{-1}(s) (b(s) + f(s, y_2(s))) ds \right| \\ &= \left| \int_0^t X(t) P_1 X^{-1}(s) (f(s, y_1(s)) - f(s, y_2(s))) ds \right. \\ &\quad \left. + \int_t^\infty X(t) P_2 X^{-1}(s) (f(s, y_2(s)) - f(s, y_1(s))) ds \right| \\ &\leq \int_0^t |X(t) P_1 X^{-1}(s)| |f(s, y_1(s)) - f(s, y_2(s))| ds \end{aligned}$$

$$\begin{aligned}
 & + \int_t^\infty |X(t) P_2 X^{-1}(s)| |f(s, y_1(s)) - f(s, y_2(s))| ds \\
 & \leq \int_0^t |X(t) P_1 X^{-1}(s)| \gamma |y_1(s) - y_2(s)| ds \\
 & + \int_t^\infty |X(t) P_2 X^{-1}(s)| \gamma |y_1(s) - y_2(s)| ds \leq |y_1 - y_2| \gamma K.
 \end{aligned}$$

We wish to prove that $|\tau y(t)| \leq \beta$. Now

$$\begin{aligned}
 |\tau y(t)| & = \left| \int_0^t X(t) P_1 X^{-1}(s) (b(s) + f(s, y(s))) ds \right. \\
 & \quad \left. - \int_t^\infty X(t) P_2 X^{-1}(s) (b(s) + f(s, y(s))) ds \right| \\
 & \leq \int_0^t |X(t) P_1 X^{-1}(s) b(s)| ds + \int_0^t |X(t) P_1 X^{-1}(s) f(s, y(s))| ds \\
 & \quad + \int_t^\infty |X(t) P_2 X^{-1}(s) b(s)| ds + \int_t^\infty |X(t) P_2 X^{-1}(s) f(s, y(s))| ds \\
 & \leq \left(\int_0^t |X(t) P_1 X^{-1}(s)|^q ds \right)^{1/q} |b|_p + \gamma \beta \int_0^t |X(t) P_1 X^{-1}(s)| ds \\
 & \quad + \left(\int_t^\infty |X(t) P_2 X^{-1}(s)|^q ds \right)^{1/q} |b|_p + \gamma \beta \int_t^\infty |X(t) P_2 X^{-1}(s)| ds.
 \end{aligned}$$

This last inequality was obtained by applying the Hölder inequality. We now conclude that

$$\begin{aligned}
 |\tau y(t)| & \leq |b|_p 2K^{1/q} + \gamma \beta K \\
 & \leq \beta.
 \end{aligned}$$

By the contraction principle, the integral equation $y = \tau y$ has a unique solution $y(t)$ such that $|y(t)| \leq \beta$. By differentiation, we see that this $y(t)$ is a solution of (4).

Let us now assume that our perturbed linear system is

$$(5) \quad y' = A(t)y + f(t, y),$$

where $|f(t, y(t))| \in L^p[0, \infty)$. Let the L^p norm of $f(t, y)$ for $|y| \leq \beta$ be denoted by $|f|_p^\beta$ and let $X(t)$, P_1 and P_2 be as previously described.

THEOREM 3. Assume that

$$\left(\int_0^t |X(t) P_1 X^{-1}(s)|^q ds \right)^{1/q} + \left(\int_t^\infty |X(t) P_2 X^{-1}(s)|^q ds \right)^{1/q} \leq K.$$

Let $x(t)$ be a solution of (3) such that $|x(t)| \leq \alpha$ for some constant α and assume that

$$|f|_p^{2\alpha} \leq \frac{\alpha}{K}.$$

Then there exists a solution $y(t)$ of (5) such that $|y(t)| \leq 2\alpha$.

Proof. For $|y(t)| \leq 2\alpha$ define the integral operator G by

$$\begin{aligned} G(y(t)) = & x(t) + \int_0^t X(t) P_1 X^{-1}(s) f(s, y(s)) ds \\ & - \int_t^\infty X(t) P_2 X^{-1}(s) f(s, y(s)) ds. \end{aligned}$$

Then

$$\begin{aligned} |G(y(t))| \leq & |x(t)| + \int_0^t |X(t) P_1 X^{-1}(s)| |f(s, y(s))| ds \\ & + \int_t^\infty |X(t) P_2 X^{-1}(s)| |f(s, y(s))| ds \\ \leq & \alpha + \left(\int_0^t |X(t) P_1 X^{-1}(s)|^q ds \right)^{1/q} |f|_p^{2\alpha} \\ & + \left(\int_t^\infty |X(t) P_2 X^{-1}(s)|^q ds \right)^{1/q} |f|_p^{2\alpha} \\ \leq & \alpha + |f|_p^{2\alpha} K \leq \alpha + \alpha = 2\alpha. \end{aligned}$$

Thus Gy is uniformly bounded for all $y(t)$ such that $|y(t)| \leq 2\alpha$.

The continuity of f can be used to establish the continuity of G .

Now Gy is a solution of

$$(6) \quad v' = A(t)v + f(t, y(t))$$

since $x(t)$ is a solution of (3). Hence Gy has a uniformly bounded derivative on finite intervals, from which it follows that $\{Gy\}$ is equicontinuous. All hypotheses of Schauder's fixed point theorem are satisfied and hence the equation $Gy = y$ has a solution $y(t)$ such that $|y(t)| \leq 2\alpha$. Because of (6), this $y(t)$ is also a solution of (5).

Remark. To handle the case when $p = 1$ we assume that

$$\begin{aligned} |X(t) P_1 X^{-1}(s)| &\leq K, & 0 \leq s \leq t, \\ |X(t) P_2 X^{-1}(s)| &\leq K, & 0 \leq t \leq s. \end{aligned}$$

The proof remains almost the same.

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