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Oscillation theorems for characteristic initial value problems for linear hyperbolic equations

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Matematica. — *Oscillation theorems for characteristic initial value problems for linear hyperbolic equations* (*). Nota (**) di GORDON PAGAN, presentata dal Socio M. PICONE.

RIASSUNTO. — In questa Nota sono stabiliti teoremi di oscillazione per soluzioni della equazione $v_{xy} + G(x, y) v = 0$ individuate da condizioni iniziali.

Tali teoremi sono estesi a equazioni del tipo

$$v_{xy} + a(x, y) v_x + b(x, y) v_y + G(x, y) v = 0 .$$

Sono anche descritti alcuni caratteri qualitativi di soluzioni delle equazioni stesse.

1. INTRODUCTION

The purpose of the paper is to study the oscillatory behaviour of solutions of the hyperbolic characteristic initial value problem

$$(1.1) \qquad \begin{aligned} & v_{xy} + a(x, y) v_x + b(x, y) v_y + g(x, y) v = 0 \\ & v(x, 0) = F(x) \quad ; \quad v(0, y) = G(y) \quad ; \quad x, y > 0 \end{aligned}$$

where it is assumed that a, b, g, F and G are continuous and that $v(x, y)$ is a C^2 solution of (1.1). We shall be especially interested in establishing criteria for the oscillatory behaviour of such solutions. A non-trivial solution $v(x, y)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros—i.e. if for every $r > 0$ there exists $x > 0$ and $y > 0$ such that $x^2 + y^2 \geq r^2$ and $v(x, y) = 0$.

In the special case of self adjoint equations, Kreith [1] established such oscillation criteria by generalising the technique used by Sturm for ordinary differential equations. While the present techniques are similar, they avoid the regularity conditions required in [1] and also extend to non-self adjoint equations. The basic techniques are presented in § 2 below and the extension to the non-self adjoint equation in § 3. Some more difficult questions about the nature of the zeros of oscillatory solutions are discussed in § 4.

2. SELF ADJOINT EQUATIONS

Consider the characteristic initial value problem

$$(2.1) \qquad \begin{aligned} & v_{xy} + g(x, y) v = 0 \\ & v(x, 0) = F(x) \quad ; \quad v(0, y) = G(y) \quad ; \quad x, y > 0 . \end{aligned}$$

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We note that $u = \sin(\mathrm{L}f(x) + y/\mathrm{L})$ is a solution of

$$u_{xy} + f'(x)\,u = 0 \tag{2.2}$$

where L is an arbitrary constant. This fact leads to the following result.

THEOREM I. *Let $f(x)$ be a differentiable monotonic increasing function satisfying $f(0) = 0$. If there exists a constant $\mathrm{L} \neq 0$ for which*

(i) $g(x, y) \geq f'(x)$;
(ii) $\mathrm{F}(x) > 0$; $\mathrm{G}(y) > 0$;
(iii) $\mathrm{F}'(x) + \mathrm{L}^2 f'(x)\,\mathrm{G}'(\mathrm{L}^2 f(x)) \leq 0$

then the solution $v(x, y)$ of (2.1) changes sign in each region

$$\mathrm{D}_n = \{(x, y) \mid x > 0, y > 0, \mathrm{L}n\pi < y + \mathrm{L}^2 f(x) < \mathrm{L}(n+1)\pi\}.$$

Proof

$$\text{Let } \mathrm{L}(v) \equiv v_{xy} + g(x, y)\,v = 0$$
$$l(u) \equiv u_{xy} + f'(x)\,u = 0.$$

Then

$$u\mathrm{L}(v) - vl(u) \equiv (g(x, y) - f'(x))\,uv + (v_x u)_y - (u_y v)_x = 0. \tag{2.3}$$

Integrating over a sufficiently smooth bounded domain D_n yields

$$\iint_{\mathrm{D}_n} (g(x, y) - f'(x))\,uv\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathrm{D}_n} ((u_y v)_x - (v_x u)_y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.4}$$

Since $u(x, y) = \sin(\mathrm{L}f(x) + y/\mathrm{L})$ is zero when $y = \mathrm{L}n\pi - \mathrm{L}^2 f(x)$, it is useful to define $\mathrm{D}_n = ((x, y)/x > 0, y > 0, \mathrm{L}n\pi < y + \mathrm{L}^2 f(x) < \mathrm{L}(n+1)\pi)$. Then using Green's theorem in (2.4) gives

$$\iint_{\mathrm{D}_n} (g(x, y) - f'(x))\,uv\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial \mathrm{D}_n} v_x u\,\mathrm{d}x + u_y v\,\mathrm{d}y. \tag{2.5}$$

Hence (2.5) becomes

$$\iint_{\mathrm{D}_n} (g(x, y) - f'(x))\,uv\,\mathrm{d}x\,\mathrm{d}y = \int_{f^{-1}(n\pi/\mathrm{L})}^{f^{-1}((n+1)\pi/\mathrm{L})} (v_x(x, 0)\sin \mathrm{L}f(x))\,\mathrm{d}x \tag{2.6}$$

$$+ \int_0^{\mathrm{L}(n+1)\pi} (1/\mathrm{L}\cos(n+1)\pi\, v(f^{-1}((n+1)\pi/\mathrm{L} - y/\mathrm{L}^2), y))\,\mathrm{d}y$$

$$- \int_{\mathrm{L}n\pi}^{\mathrm{L}(n+1)\pi} (1/\mathrm{L}\cos(y/\mathrm{L})\,v(0, y))\,\mathrm{d}y - \int_0^{\mathrm{L}n\pi} (1/\mathrm{L}\cos n\pi\, v(f^{-1}(n\pi/\mathrm{L} - y/\mathrm{L}^2), y))\,\mathrm{d}y.$$

Now

$$\int_{L n\pi}^{L(n+1)\pi} (1/L \cos (y/L)\, v(0, y))\, dy = -\int_{L n\pi}^{L(n+1)\pi} (\sin (y/L)\, v_y(0, y))\, dy.$$

Substituting into (2.6) we get

$$(2.7) \qquad \iint_{D_n} (g(x, y) - f'(x))\, uv\, dx\, dy = \int_{f^{-1}(n\pi/L)}^{f^{-1}((n+1)\pi/L)} F'(x) \sin Lf(x)\, dx$$

$$+ \int_{L n\pi}^{L(n+1)\pi} (\sin (y/L)\, G'(y))\, dy$$

$$+ \int_{0}^{L(n+1)\pi} (1/L \cos (n+1)\pi\, v(f^{-1}((n+1)\pi/L - y/L^2), y))\, dy$$

$$- \int_{0}^{L n\pi} (1/L \cos n\pi\, v(f^{-1}(n\pi/L - y/L^2), y))\, dy.$$

Substituting $y = L^2 f(x)$ in the second integral on the right hand side of (2.7) yields

$$(2.8) \qquad \iint_{D_n} (g(x, y) - f'(x))\, uv\, dx\, dy =$$

$$\int_{f^{-1}(n\pi/L)}^{f^{-1}((n+1)\pi/L)} \sin (Lf(x))\, (F'(x) + L^2 f'(x)\, G'(L^2 f(x)))\, dx$$

$$+ \int_{0}^{L(n+1)\pi} 1/L \cos (n+1)\pi\, v(f^{-1}((n+1)\pi/L - y/L^2), y)\, dy$$

$$- \int_{0}^{L n\pi} 1/L \cos n\pi\, v(f^{-1}(n\pi/L - y/L^2), y)\, dy.$$

If $v(x, y)$ does not change sign in D_n, then $v(x, y)$ is positive in D_n by (ii).

If n is even, the hypotheses of the theorem assure that the left hand side of (2.8) is non positive, while the right hand side is positive. This contradiction shows that $v(x, y)$ changes sign in each D_n. A similar argument applies for n odd. Q.E.D.

COROLLARY. *If there exists a constant* $L \neq 0$ *such that*

(i) $g(x, y) \geq k^2$ (*k a constant*);

(ii) $F(x) > 0$; $G(y) > 0$;

(iii) $F'(x) + L^2 k^2 G'(L^2 k^2 x) \leq 0$

then every solution v of (2.1) *changes sign in each region*

$$D_n = \{(x, y) \mid x > 0 ; y > 0 , Ln\pi < y + L^2 k^2 x < L(n + 1)\pi\}.$$

Proof. Set $f(x) = k^2 x$ in the theorem.

3. NON SELF ADJOINT EQUATIONS

We first prove a comparison theorem for the special non self adjoint equation

$$v_{xy} + a(x, y)(v_x - v_y) + g(x, y) v = 0 \tag{3.1}$$
$$v(x, 0) = F(x) \quad ; \quad v(0, y) = G(y).$$

THEOREM II. *If there exists a positive constant* k^2 *such that*

(i) $g(x, y) \geq a_x(x, y) - a_y(x, y) + k^2$

(ii) $F(x) > 0$; $G(y) > 0$

(iii) $F'(x) + G'(x) + a(0, x) G(x) - a(x, 0) F(x) \leq 0$

then every solution $v(x, y)$ *of* (3.1) *changes sign in each domain*

$$D_n = \{(x, y) \mid x > 0 ; y > 0 ; n\pi/k < x + y < (n + 1)\pi/k\}.$$

Proof. We note that $u = \sin k(x + y)$ is a solution of

$$u_{xy} + k^2 u = 0 . \tag{3.2}$$

Multiplying (3.2) by $v(x, y)$ and (3.1) and subtracting yields

$$(g(x, y) - k^2) uv = (vu_y)_x - (uv_x)_y - av_x u + av_y u. \tag{3.3}$$

Hence

$$(g(x, y) - a_x + a_y - k^2) uv = (vu_y - auv)_x - (uv_x - auv)_y + a(u_x - u_y) v. \tag{3.4}$$

For $u = \sin k(x + y)$ we have $u_x = u_y$ and the last term of (3.4) is zero. Integrating over a domain

$$D_n = \{(x, y) | x > 0 ; y > 0 ; n\pi/k < x + y < (n + 1)\pi/k\}$$

yields

$$\iint_{D_n} (g - a_x + a_y - k^2) uv \, dx \, dy = \iint_{D_n} ((vu_y - auv)_x - (uv_x - auv)_y) \, dx \, dy. \tag{3.5}$$

Using Green's theorem on (3.5) gives

$$(3.6)\qquad \iint_{D_n} (g - a_x + a_y - k^2)\, uv \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial D_n} (vu_y - auv)\,\mathrm{d}y + (uv_x - auv)\,\mathrm{d}x.$$

Hence (3.6) becomes explicity

$$(3.7)\qquad \iint_{D_n} (g - a_x + a_y - k^2)\, uv \,\mathrm{d}x\,\mathrm{d}y =$$

$$\int_{n\pi/k}^{(n+1)\pi/k} \sin kx\,(v_x(x\,,\,0) - a(x\,,\,0)\,v(x\,,\,0))\,\mathrm{d}x$$

$$+ \int_0^{(n+1)\pi/k} kv\,(-y + (n+1)\,\pi/k\,,\,y)\cos(n+1)\,\pi\,\mathrm{d}y$$

$$- \int_{n\pi/k}^{(n+1)\pi/k} k\cos ky\, v(0\,,\,y) - a(0\,,\,y)\sin ky\, v(0\,,\,y)\,\mathrm{d}y$$

$$- \int_0^{n\pi/k} kv\,(-y + n/k\,,\,y)\cos n\pi\,\mathrm{d}y.$$

Noting that

$$(3.8)\qquad -\int_{n\pi/k}^{(n+1)\pi/k} (k\cos ky\, v(0\,,\,y))\,\mathrm{d}y = \int_{n\pi/k}^{(n+1)\pi/k} (\sin ky\, v_y(0\,,\,y))\,\mathrm{d}y$$

and substituting (3.8) into the first part of the third integral on the right hand side of (3.7) yields

$$(3.9)\qquad \iint_{D_n} (g - a_x + a_y - k^2)\, uv\,\mathrm{d}x\,\mathrm{d}y = \int_{n\pi/k}^{(n+1)\pi/k} \sin kx\,(F'(x) - a(x\,,0)\,F(x))\,\mathrm{d}x$$

$$+ \int_0^{(n+1)\pi/k} kv\,(-y + (n+1)\,\pi/k\,,\,y)\cos(n+1)\,\pi\,\mathrm{d}y$$

$$+ \int_{n\pi/k}^{(n+1)\pi/k} \sin ky\,(G'(y) + a(0\,,\,y)\,G(y))\,\mathrm{d}y$$

$$- \int_0^{n\pi/k} kv\,(-y + n\pi/k\,,\,y)\cos n\pi\,\mathrm{d}y.$$

Finally putting $y = x$ in the third integral yields

$$\iint_{D_n} (g - a_x + a_y - k^2)\, uv\, dx\, dy = \tag{3.10}$$

$$\int_{n\pi/k}^{(n+1)\pi/k} \sin kx\, (F'(x) + G'(x) + a(0, x)\, G(x) - a(x, 0)\, F(x))\, dx$$

$$+ \int_0^{(n+1)\pi/k} kv(-y + (n+1)\pi/k, y) \cos(n+1)\pi\, dy$$

$$- \int_0^{n\pi/k} kv(-y + n\pi/k, y) \cos n\pi\, dy.$$

Assuming $v(x, y)$ does not change sign in D_n, it follows from condition (ii) that $v(x, y) > 0$. If n is even, the hypotheses of the theorem assure that the left hand side of (3.10) is non positive, while the right hand side is positive. This contradiction shows that $v(x, y)$ changes sign in each D_n. A similar argument applies for n odd. Q.E.D.

To deal with the general case it is necessary to prove the following lemma.

LEMMA. *If v satisfies $v_{xy} + a(x, y)\, v_x + b(x, y)\, v_y + c(x, y)\, v = 0$ and if* $Z(x, y) = v(x, y) \exp\left(\frac{1}{2} \int_0^s (\tilde{a}(\alpha, t) + \tilde{b}(\alpha, t))\, d\alpha\right)$ *where* $\tilde{a}(s, t) = a(x, y)$; $\tilde{b}(s, t) = b(x, y)$; $x + y = s$ *and* $x - y = t$, *then* Z *satisfies*

$$Z_{xy} + p(x, y)(Z_x - Z_y) + q(x, y)\, Z = 0$$

where

$$p(x, y) = \frac{1}{2}\left(a(x, y) - b(x, y) + \int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right)$$

and

$$q(x, y) = \frac{1}{2}\left(\int_0^s (\tilde{a}_{tt}(\alpha, t) + \tilde{b}_{tt}(\alpha, t))\, d\alpha - \frac{1}{2}(a_x + a_y + b_x + b_y)\right)$$

$$- \frac{1}{4}\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right)^2 - \frac{1}{4}(a + b)^2$$

$$+ \frac{1}{2}(b - a)\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right) + c(x, y).$$

Proof. Consider

$$v_{xy} + av_x + bv_x + cv = 0 \tag{3.11}$$

and

$$v = w(x, y) Z(x, y) \tag{3.12}$$

where

$$w(x, y) = \tilde{w}(s, t) = \exp\left(-\frac{1}{2}\int_0^s (\tilde{a}(\alpha, t) + \tilde{b}(\alpha, t))\, d\alpha\right). \tag{3.13}$$

Substituting (3.12) into (3.11) we get

$$\begin{gathered} Z_{xy} + (w_y/w + a) Z_x + (w_x/w + b) Z_y \\ + (w_{xy}/w + aw_x/w + bw_y/w + c) Z = 0. \end{gathered} \tag{3.14}$$

Hence

$$w_y/w = \left(-\frac{1}{2}(\tilde{a}(s, t) + \tilde{b}(s, t)) + \frac{1}{2}\int_0^s (\tilde{a}(\alpha, t) + \tilde{b}(\alpha, t))\, d\alpha\right) \tag{3.15}$$

and

$$w_x/w = \left(-\frac{1}{2}(\tilde{a}(s, t) + \tilde{b}(s, t)) - \frac{1}{2}\int_0^s (\tilde{a}(\alpha, t) + \tilde{b}(\alpha, t))\, d\alpha\right). \tag{3.16}$$

So that

$$(w_x/w + b) = -(w_y/w + a)$$

and

$$p(x, y) = \frac{1}{2}\left(a - b + \int_0^s (\tilde{a}(\alpha, t) + \tilde{b}(\alpha, t))\, d\alpha\right). \tag{3.17}$$

Differentiating (3.16) with respect to y we get

$$\begin{gathered} w_{xy}/w = -\left(\frac{1}{4}\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right)^2 - \frac{1}{4}(a + b)^2\right) \\ -\frac{1}{2}\left(a_y + b_y + \tilde{a}_t(s, t) + \tilde{b}_t(s, t) - \int_0^s (\tilde{a}_{tt}(\alpha, t) + \tilde{b}_{tt}(\alpha, t))\, d\alpha\right). \end{gathered} \tag{3.18}$$

Hence multiplying (3.15) by $a(x, y)$ and (3.16) by $b(x, y)$ yields

$$w_{xy}/w + aw_x/w + bw_y/w + c = \tag{3.19}$$

$$\frac{1}{2}\left(\int_0^s (\tilde{a}_{tt}(\alpha, t) + \tilde{b}_{tt}(\alpha, t))\, d\alpha - \frac{1}{2}(a_x + a_y + b_x + b_y)\right)$$

$$-\frac{1}{4}\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right)^2 - \frac{1}{4}(a+b)^2$$

$$+\frac{1}{2}(b-a)\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right) + c(x, y)$$

which is the required form $q(x, y)$ in (3.14).

Hence our new equation is of the form

$$Z_{xy} + p(x, y)(Z_x - Z_y) + q(x, y)Z = 0 \qquad \text{Q.E.D.}$$

Remark. Since $Z(x, y) = 0$ if and only if $v(x, y) = 0$, the transformation used in the lemma preserves oscillation properties. This fact will be useful in the next theorem.

THEOREM III. *Let v be a solution of* (1.1) *where the following conditions hold*:

(i) $$g(x, y) \geq \frac{1}{2}\int_0^s (\tilde{a}_{tt}(\alpha, t) + \tilde{b}_{tt}(\alpha, t))\, d\alpha$$

$$+\frac{1}{4}(3a_x - b_x - a_y + 3b_y) + \frac{1}{4}\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right)^2$$

$$+\frac{1}{2}(a-b)\left(\int_0^s (\tilde{a}_t(\alpha, t) + b_t(\alpha, t))\, d\alpha\right) + \frac{1}{4}(a+b)^2 + k^2$$

where k is a constant and $t = x - y$; $s = x + y$; $\tilde{a}(s, t) = a(x, y)$; $\tilde{b}(s, t) = b(x, y)$

(ii) $F(x) > 0$; $G(y) > 0$

(iii) $$\exp\left(\frac{1}{2}\int_0^x (\tilde{a}(\alpha, x) + \tilde{b}(\alpha, x))\, d\alpha\right)(F'(x) + b(x, 0)F(x))$$

$$+\exp\left(\frac{1}{2}\int_0^x (\tilde{a}(\alpha, -x) + \tilde{b}(\alpha, -x))\, d\alpha\right)(G'(x) + a(0, x)G(x)) \leq 0$$

then $v(x, y)$ *changes sign in each* D_n

$$D_n = \{(x, y) \mid x > 0, y > 0; n\pi/k < x + y < (n+1)\pi/k\}.$$

Proof. Consider $v_{xy} + a(x, y) v_x + b(x, y) v_y + g(x, y) v = 0$ and

$$Z = v \exp\left(\frac{1}{2}\int_0^s (\tilde{a}(\alpha, t) + \tilde{b}(\alpha, t))\, d\alpha\right).$$

Then

$$Z_{xy} + p(Z_x - Z_y) + qZ = 0 \tag{3.20}$$

where p and q are defined in the preceding lemma.

By condition (i) in the theorem

$$g(x, y) \geq \frac{1}{2}\int_0^s (\tilde{a}_{tt}(\alpha, t) + \tilde{b}_{tt}(\alpha, t))\, d\alpha \tag{3.21}$$

$$+ \frac{1}{4}(3a_x - b_x - a_y + 3b_y) + \frac{1}{4}\left(\int_0^s (\tilde{a}_t(\alpha, t) + \tilde{b}_t(\alpha, t))\, d\alpha\right)^2$$

$$+ \frac{1}{2}(a - b)\left(\int_0^s (\tilde{a}_t + \tilde{b}_t)\, d\alpha\right) + \frac{1}{4}(a + b)^2 + k^2,$$

so that a rearrangement of terms in (3.21) yields

$$\frac{1}{2}\left(\int_0^s (\tilde{a}_{tt} + \tilde{b}_{tt})\, d\alpha - \frac{1}{2}(a_x + a_y + b_x + b_y)\right) \tag{3.22}$$

$$- \frac{1}{4}\left(\int_0^s (\tilde{a}_t + \tilde{b}_t)\, d\alpha\right)^2 - \frac{1}{4}(a + b)^2$$

$$+ \frac{1}{2}(b - a)\left(\int_0^s (\tilde{a}_t + \tilde{b}_t)\, d\alpha\right) + g(x, y) \geq$$

$$\int_0^s (\tilde{a}_{tt} + \tilde{b}_{tt})\, d\alpha + \frac{1}{2}(a_x - b_x - a_y + b_y) + k^2.$$

The left hand side of (3.22) is $q(x, y)$ of (3.20). The right hand side of (3.22) is $p_x - p_y + k^2$, where $p(x, y)$ is the coefficient appearing in (3.20)

Hence condition (i) leads to

$$q(x, y) \geq p_x(x, y) - p_y(x, y) + k^2. \tag{3.23}$$

Using the conditions $F(x) > 0$; $G(y) > 0$, and our transformation we have

$$F(x) > 0 \quad ; \quad G(y) > 0 \tag{3.24}$$

if and only if

$$Z(x, 0) > 0 \quad ; \quad Z(0, y) > 0.$$

Finally, by condition (iii)

$$(F'(x) + b(x, 0) F(x)) \left(\exp \frac{1}{2} \int_0^x (\tilde{a}(\alpha, x) + \tilde{b}(\alpha, x)) \, d\alpha\right) \tag{3.25}$$

$$+ (G'(x) + a(0, x) G(x)) \left(\exp \frac{1}{2} \int_0^x (\tilde{a}(\alpha, -x) + \tilde{b}(\alpha, -x)) \, d\alpha\right) \leq 0.$$

By our transformation

$$Z(x, 0) = F(x) \exp \left(\frac{1}{2} \int_0^x (\tilde{a}(\alpha, x) + \tilde{b}(\alpha, x)) \, d\alpha\right) \tag{3.26}$$

and

$$Z(0, x) = G(x) \exp \left(\frac{1}{2} \int_0^x (\tilde{a}(\alpha, -x) + \tilde{b}(\alpha, -x)) \, d\alpha\right) \tag{3.27}$$

i.e.

$$F(x) = Z(x, 0) \exp \left(-\frac{1}{2} \int_0^x (\tilde{a}(\alpha, x) + \tilde{b}(\alpha, x)) \, d\alpha\right) \tag{3.28}$$

$$G(x) = Z(0, x) \exp \left(-\frac{1}{2} \int_0^x (\tilde{a}(\alpha, -x) + \tilde{b}(\alpha, -x)) \, d\alpha\right). \tag{3.29}$$

Hence

$$F'(x) = \left(Z_x(x, 0) - \frac{1}{2} \left(a(x, 0) + b(x, 0) \right.\right. \tag{3.30}$$

$$\left.\left. + \int_0^x (\tilde{a}_x(\alpha, x) + \tilde{b}_x(\alpha, x)) \, d\alpha\right) Z(x, 0)\right) \cdot$$

$$\cdot \exp \left(-\frac{1}{2} \int_0^x (\tilde{a}(\alpha, x) + \tilde{b}(\alpha, x)) \, d\alpha\right)$$

$$(3.31) \qquad G'(x) = \Bigg(Z_x(0, x) - \frac{1}{2}\Bigg(a(0, x) + b(0, x) - \int_0^x (\tilde{a}_x(\alpha, -x) + \tilde{b}_x(\alpha, -x))\, d\alpha\Bigg) Z(0, x)\Bigg) \cdot \exp\Bigg(-\frac{1}{2}\int_0^x (\tilde{a}(\alpha, -x) + \tilde{b}(\alpha, -x))\, d\alpha\Bigg).$$

Multiplying (3.28) by $b(x, 0)$ and (3.29) by $a(0, x)$ then substituting with (3.30) and (3.31) into (3.25) we have

$$Z_x(x, 0) - \frac{1}{2}\Bigg((a(x, 0) - b(x, 0)) + \int_0^x (\tilde{a}_x(\alpha, x) + \tilde{b}_x(\alpha, x))\, d\alpha\Bigg) Z(x, 0) + Z_x(0, x) + \frac{1}{2}\Bigg(a(0, x) - b(0, x) + \int_0^x (\tilde{a}_x(\alpha, -x) + \tilde{b}_x(\alpha, -x))\, d\alpha\Bigg) Z(0, x) \leq 0$$

which is

$$(3.32) \qquad Z_x(x, 0) - p(x, 0) Z(x, 0) + Z_x(0, x) + p(0, x) Z(0, x) \leq 0.$$

By Theorem II, the solution of (3.20) satisfying (3.23), (3.24) and (3.32) changes sign in each

$$D_n = \{(x, y) \mid x > 0 ; y > 0 ; n\pi/k < x + y < (n+1)\pi/k\}.$$

By the remark preceding this theorem, $v(x, y)$ changes sign in each D_n.

Q.E.D.

COROLLARY. *If v is a solution of* (1.1) *where a and b are constants and the following conditions hold*

(i) $g(x, y) \geq \frac{1}{4}(a+b)^2 + k^2$ *where k is a constant*

(ii) $F(x) > 0$; $G(y) > 0$

(iii) $F'(x) + bF(x) + G'(x) + aG(x) \leq 0$

then $v(x, y)$ changes sign in each D_n

$$D_n = \{(x, y) \mid x > 0 ; y > 0 ; n\pi/k < x + y < (n+1)\pi/k\}.$$

4. Nature of oscillatory solutions

Certain questions can be posed as to the nature of oscillatory solutions of (2.1). For example:

(i) Are there any criteria to assure that $v(x, y) = 0$ for some (x, y) satisfying $x > m$; $y > m$ where m is arbitrarily large?

(ii) If $v(x, y) = 0$ on a nodal curve given to be $y = f(x)$, what is the nature of $f(x)$?

(iii) Is there an infinite number of nodal curves?

The following results, while not dealing with all aspects of their behaviour, do seem to give further insight into the qualitative nature of solutions of (2.1).

THEOREM IV. *Let v be a solution of* (2.1) *where*

(i) $g(x, y) \geq k^2$; *k a constant*;

(ii) $F(x) > 0$; $G(y) > 0$;

(iii) $F'(x) \leq 0$; $G'(y) \leq 0$.

Then for every positive constant L, *v changes sign in each region*

$$D_n(L) = \{(x, y) | x > 0 ; y > 0 ; Ln\pi < y + L^2 k^2 x < L(n+1)\pi\}.$$

Proof. This follows directly from Theorem I.

Remark. It follows that $v(x, y)$ must change sign in this domain for arbitrary values of L. Hence $v(x, y)$ must change sign arbitrarily close to both axes.

THEOREM V. *If v is a solution of* (2.1) *where*

(i) $g(x, y) \geq k^2$; *k is a constant*;

(ii) $F(x) > 0$; $G(y) > 0$;

(iii) $F'(x) \leq 0$; $G'(y) \leq 0$;

and $f(x)$ is a C^2 function satisfying $v(x, f(x)) = 0$, $v(x, y) > 0$ for $0 < y < f(x)$. Then $f'(x) < 0$.

Proof. Let $v(x, y) = 0$ on the line $y = f(x)$, and let $y = f(x)$ be increasing between $x = a$ and $x = b$ so that $v_x(x, f(x)) \geq 0$ between $x = a$ and $x = b$.

Considering $v_{xy} + gv = 0$.

Green's theorem yields

$$\iint_D gv \, dx \, dy = \oint_{\partial D} v_x \, dx \tag{4.1}$$

where $D = ((x, y) / a < x < b ; y < f(x))$.

Hence (4.1) becomes explicity

$$\iint_D gv \, dx \, dy - \int_a^b F'(x) \, dx = - \int_a^b v_x(x, f(x)) \, dx. \tag{4.2}$$

By conditions (i) and (iii), (4.2) gives

$$\int_a^b v_x(x, f(x)) \, dx < 0.$$

Which contradicts $v_x(x, f(x)) \geq 0$. Hence $y = f(x)$ is monotonic decreasing.

Remark. By the remark preceding this theorem, the nodal curves of $v(x, y)$ are arbitrarily close to the axes. Because of the monotonicity of $f(x)$ in Theorem V, these nodal curves are asymptotic to the axes.

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REFERENCES

[1] K. KREITH, *Sturmian theorems for characteristic Initial Value Problems*, « Rendiconti della Accademia Nazionale dei Lincei », ser. VIII, *47* (3-4), 139-144 (1969).