Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Gheorghe Bocșan, Gheorghe Constantin

The Kuratowski function and some applications to the probabilistic metric spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **55** (1973), n.3-4, p. 236–240.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_55_3-4_236_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma
bdim (Biblioteca Digitale Italiana di Matematica)
SIMAI & UMI
http://www.bdim.eu/

Calcolo delle probabilità. — *The Kuratowski function and some applications to the probabilistic metric spaces.* Nota [(*)] di GHEORGHE BOCŞAN e GHEORGHE CONSTANTIN, presentata dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori estendono il numero di Kuratowski allo spazio metrico probabilistico introdotto da K. Menger e danno alcune caratterizzazioni degli insiemi probabilistici precompatti.

1. The compactness in topological spaces offers techniques for the proof of certain basic theorems in analysis. In order to generalize the compactness the notion of measure of noncompactness was introduced. Actually there are two such measures one introduced by Kuratowski and one by Hausdorff.

Let (X, d) be a complete metric space and A a bounded subset of X. It is known that the Kuratowski's number $\alpha(A)$ is the greatest lower bound of the set of positive real numbers ε such that A can be covered by a finite family of subsets of X whose diameters are less than or equal to ε [6]. It is of interest to emphasize that this concept is the origin of a great number of recent generalizations of some fixed points theorems [1], [3], [4], [5], [9], [12].

Our purpose in this Note is to generalize this notion to probabilistic metric spaces introduced by K. Menger [7], to define the precompactness and to give some important properties.

2. Let Δ be the set of nondecreasing, left continuous functions $F : R \to [0, 1]$ such that $F(0) = 0$ and $\lim_{x \to \infty} F(x) = 1$. We recall that a probabilistic metric space is an ordered pair (S, F) where S is an arbitrary set and F is a Δ–valued function defined on $S \times S$. In what follows $F(p, q)$ will be denoted by F_{pq} [7], [9].

DEFINITION 2.1. [2]. *Let* A *be a nonempty subset of* S. *The function* $D_A(\cdot)$ *defined by*

$$D_A(x) = \sup_{t < x} \inf_{p,q \in A} F_{pq}(t)$$

is called the probabilistic diameter of A. A *is called bounded if* $\sup_x D_A(x) = 1$.

DEFINITION. 2.2. *Let* A *be a bounded subset of* S. *The mapping* $\alpha_A(x) = \sup \{\varepsilon \geq 0 : \exists \{A_j\}_{j \in J},\ J = \textit{finite},\ A = \bigcup_{j \in J} A_j,\ D_{A_j}(\varepsilon) \geq \varepsilon\}$ *is called the Kuratowski's probabilistic number or Kuratowski's function.*

(*) Pervenuta all'Accademia il 25 settembre 1973.

PROPOSITION 2.1. *The Kuratowski's function has the following properties*

1) $\alpha_A \in \Delta$,

2) $\alpha_A(x) \geq D_A(x)$,

3) *if* $\varnothing \neq A \subset B \subset S$ *then* $\alpha_A(x) \geq \alpha_B(x)$,

4) $\alpha_{A \cup B}(x) = \min\{\alpha_A(x), \alpha_B(x)\}$,

5) *let* $\bar{A}$ *be the closure of* A *in the* (ε, λ)*-topology of* S.

Then $\alpha_A(x) = \alpha_{\bar{A}}(x)$.

Proof. 1) Clearly, if $x = 0$, $D_A(0) = 0$ for every $A \subset S$. Therefore $\alpha_A(0) = 0$. To prove that $\alpha_A(\cdot)$ is nondecreasing, let us observe that if $x_1 < x_2$ then $D_{A_j}(x_1) \leq D_{A_j}(x_2)$ for each $j \in J$ which implies that $\alpha_A(x_1) \leq \alpha_A(x_2)$.

Since A is bounded then there exists a partition $\{A_j\}_{j \in J}$ such that $\lim_{x \to \infty} D_{A_j}(x) = 1$. Thus for every $\eta \in (0, 1)$ there exists x_j such that $D_{A_j}(x_j) \geq 1 - \eta$. If $x = \max_{j \in J} x_j$ then $D_{A_j}((x, \infty)) \subset [1 - \eta, 1]$ which implies that $\lim_{x \to \infty} \alpha_A(x) = 1$.

It remains to prove that $\alpha_A(\cdot)$ is left continuous. For this let $K_{A,x} = \{\varepsilon \geq 0 : \exists \{A_j\}_{j \in J},\ J = \text{finite},\ A = \bigcup_{j \in J} A_j,\ D_{A_j}(x) \geq \varepsilon\}$, $x_0 \in (0, \infty)$ and $\varepsilon \in K_{A,x_0}$. If $\eta > 0$ then by the left continuity of D_{A_j} there exists $\delta_j > 0$ such that $D_{A_j}(x_0) - D_{A_j}(x) < \eta$ if $0 < x_0 - x < \delta_j$.

Let $\delta = \min_{j \in J} \delta_j$ and suppose that $\eta \leq \varepsilon$. If $0 < x_0 - x < \delta$ then $D_{A_j}(x) > \varepsilon - \eta$. Thus $\varepsilon - \eta \in K_{A,x}$ which implies that $\varepsilon - \eta \leq \alpha_A(x)$. Hence $\alpha_A(x_0) - \alpha_A(x) \leq \eta$ for every $x \in (x_0 - \delta, x_0)$ which proves the left continuity of α_A.

To prove 2) let $x \in R$ and $\varepsilon \geq 0$ such that $D_A(x) \geq \varepsilon$. Since $\{A\}$ is a cover of A then $\varepsilon \in K_{A,x}$ and therefore $\alpha_A(x) \geq D_A(x)$.

3) is obvious by the inclusion $K_{A,x} \supseteq K_{B,x}$ for every $x \in R$.

Since by 3) $\alpha_A(x) \geq \alpha_{A \cup B}(x)$ and $\alpha_B(x) \geq \alpha_{A \cup B}(x)$ for the proof of 4) it suffices to show that $\min\{\alpha_A(x), \alpha_B(x)\} \leq \alpha_{A \cup B}(x)$. By the Definition 2.2 for every $\delta > 0$ there exists $\varepsilon_\delta \in K_{A,x}$ and a cover $\{A_j^\delta\}_{j \in J}$ of A such that $\alpha_A(x) - \delta \leq \varepsilon_\delta \leq D_{A_j^\delta}(x)$. Similarly, there exists $\eta_\delta \in K_{B,x}$ and $\{B_i^\delta\}$ such that $\alpha_B(x) - \delta < \eta_\delta < D_{B_i^\delta}(x)$. Let us suppose that $\alpha_A(x) \leq \alpha_B(x)$. Then $\alpha_A(x) - \delta \leq \alpha_B(x) - \delta \leq D_{B_i^\delta}(x)$. Thus we obtain a cover $\{C_m\}_{m \in M}$ of $A \cup B$ such that $D_{C_m}(x) \geq \alpha_A(x) - \delta$. Therefore $\alpha_A(x) - \delta \in K_{A \cup B, x}$ which implies that $\alpha_A(x) \leq \alpha_{A \cup B}(x)$ and the property 4) is proved.

5) Since $A \subseteq \bar{A}$ it follows that $\alpha_A(x) \geq \alpha_{\bar{A}}(x)$ and for the converse inequality, we observe that if $\varepsilon \in K_{A,x}$, then there exists $\{A_i\}_{i \in I}$ with the property $A \subset \bigcup A_i$ and $D_{A_i}(x) \geq \varepsilon$. Also since $\bar{A} \subset \bigcup \bar{A}_i$ and $D_{\bar{A}_i}(x) = D_{A_i}(x) \geq \varepsilon$ [2], it follows that $\varepsilon \in K_{\bar{A},x}$ which implies that $\alpha_A(x) \leq \alpha_{\bar{A}}(x)$ and the proposition is proved.

PROPOSITION 2.2. *Let* K_A *be the set of functions* $F \in \Delta$ *such that there exists a finite cover of* A, $A = \bigcup_{j \in J} A_j$, J = *finite, with* $D_{A_j}(x) \geq F(x)$ *for every* $j \in J$. *If* $\beta_A(x) = \sup_{F \in K_A} \{F(x)\}$ *then* $\beta_A(x) = \alpha_A(x)$.

Proof. If $F \in K_A$ then $F(x) \in K_{A,x}$. Thus $F(x) \leq \alpha_A(x)$ which implies that $\beta_A(x) \leq \alpha_A(x)$.

To prove that $\alpha_A(x) \leq \beta_A(x)$ let $\varepsilon > 0$, $x > 0$. By the definition of $\alpha_A(x)$ there exists $\varepsilon' \in K_{A,x}$ such that $\alpha_A(x) - \varepsilon < \varepsilon'$. Since $\varepsilon' \in K_{A,x}$ then there exists A_j, $j = 1, 2, \cdots$, such that $D_{A_j}(x) \geq \varepsilon'$. Let $F_{x,\varepsilon'}(y) = \inf_{1 \leq j \leq n} D_{A_j}(y)$. Clearly $F_{x,\varepsilon'} \in K_A$ which implies that $F_{x,\varepsilon'}(y) \leq \beta_A(y)$. Since $D_{A_j}(x) \geq \varepsilon'$ then $F_{x,\varepsilon'}(x) \geq \varepsilon'$. Hence $\alpha_A(x) - \varepsilon < \varepsilon' \leq F_{x,\varepsilon'}(x) \leq \beta_A(x)$. Thus $\alpha_A(x) - \varepsilon \leq \beta_A(x)$ for every $\varepsilon > 0$, which proves the proposition.

It is known that the Kuratowski number in a metric space X defines a measure of noncompactness of a subset $A \subset X$. For the probabilistic metric spaces we give the following.

DEFINITION 2.3. *We say that a probabilistic metric space* S *is a probabilistic precompact space if for every* $\varepsilon > 0$, $\lambda > 0$ *there exists a finite cover of* S, $S = \bigcup_{i \in I} A_i$, I = *finite, such that* $D_{A_i}(\varepsilon) > 1 - \lambda$.

THEOREM 2.1. *If* S *is a probabilistic precompact space then for every* $\varepsilon, \lambda > 0$ *there exists a finite subset* $A_{\varepsilon\lambda}$ *of* S *such that for each* $p \in S$ *there exists* $q = q(p) \in A_{\varepsilon\lambda}$ *such that* $F_{pq}(\varepsilon) > 1 - \lambda$.

Proof. Let $D'_A(x) = \inf \{F_{pq}(x) : p, q \in A\}$. Then clearly we have $D_A(x_0) = \lim_{\substack{x \to x_0 \\ x < x_0}} D'_A(x) \leq D'_A(x_0) \leq F_{pq}(x_0)$. If we take $\varepsilon = 1/n$, $\lambda = 1/m$ then we obtain a finite cover of S, $S = \bigcup_{i \in I} A_i^{mn}$ such that $D_{A_i^{mn}}\left(\frac{1}{n}\right) > 1 - \frac{1}{m}$ which implies that $F_{pq}\left(\frac{1}{n}\right) > 1 - \frac{1}{m}$ for every $p, q \in A_i^{mn}$. Pick a point $p_i^{mn} \in A_i^{mn}$ and let $A_{mn} = \{p_i^{mn}\}_{i \in I}$. If $r \in S$ then $r \in A_i^{mn}$ and therefore $F_{rp_i^{mn}}\left(\frac{1}{n}\right) > 1 - \frac{1}{m}$.

Finally let $\varepsilon, \lambda > 0$ and $n, m \in N$ such that $1/n < \varepsilon$, $1/m < \lambda$. The set $A_{\varepsilon\lambda} = A_{mn}$ is seen to satisfy the theorem.

In the case of Menger spaces we can prove the converse of Theorem 2.1.

THEOREM 2.2. *Let* S *be a Menger space with the t-norm* T *such that* $\sup_{x<1} T(x, x) = 1$. *If for every* $\varepsilon, \lambda > 0$ *there exists a finite subset* $A_{\varepsilon\lambda}$ *of* S *as in Theorem 2.1 then* S *is a probabilistic precompact space.*

Proof. Let $A_{\varepsilon\lambda} = \{p_i\}_{i \in I}$ and $p \in S$. Then by hypothesis there exists $q \in A_{\varepsilon\lambda}$ such that $F_{pq}(\varepsilon) > 1 - \lambda$. Let $\varepsilon' = \varepsilon/2$ and λ' such that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$. Let

$$S_p(\varepsilon, \lambda) = \{q \in S : F_{pq}(\varepsilon) > 1 - \lambda\}, \qquad \varepsilon > 0, \quad \lambda > 0.$$

Clearly $S = \bigcup_{i \in I} S_{p_i}(\varepsilon', \lambda')$. It remains to prove that $D_{A_i}(\varepsilon) > 1 - \lambda$ where

$A_i = S_{p_i}(\varepsilon', \lambda')$. If $r, q \in A_i$, then $F_{rq}(\varepsilon) \geq T(F_{rp_i}(\varepsilon/2), F_{p_iq}(\varepsilon/2)) \geq$ $\geq T(1-\lambda', 1-\lambda')$ which implies that $D_{A_i}(\varepsilon', \lambda') \geq T(1-\lambda', 1-\lambda') > 1-\lambda$.

It is known that if (S, F) is a Menger space with the t-norm T such that $\sup_{x<1} T(x, x) = 1$ then the family U of the sets $U(\varepsilon, \lambda) = \{(p, q) \in S \times S$ such that $F_{pq}(\varepsilon) > 1 - \lambda\}$ is a basis of a uniformity ϑ, on S. We give the following

THEOREM 2.3. S *is a probabilistic precompact space if and only if* S *is precompact relatively to the uniformity* ϑ.

Proof. We note that S is precompact relatively to ϑ if and only if for every $\varepsilon > 0$, $\lambda > 0$ there exists $A_{\varepsilon\lambda} = \{p_1, \cdots, p_n\}$ such that $S = \bigcup_{i=1}^{n} S_{p_i}(\varepsilon, \lambda)$.

Let $\varepsilon > 0$, $\lambda > 0$ and take $\lambda' > 0$, $\eta > 0$ such that $T(1-\eta, 1-\eta) >$ $> 1 - \lambda' > 1 - \lambda$. Then there exists $p_1, \cdots, p_n$ such that $S = \bigcup_{i=1}^{n} S_{p_i}(\varepsilon/4, \eta)$.

We will show that $D_{A_i}(\varepsilon) \geq 1 - \lambda'$ where $A_i = S_{p_i}(\varepsilon/4, \eta)$. By the definition $D_{A_i}(\varepsilon) = \sup_{t<\varepsilon} \inf_{p,q \in A_i} F_{pq}(t) = \sup_{\varepsilon/2<t<\varepsilon} \inf_{p,q\in A_i} F_{pq}(t)$. But $F_{pq}(t) \geq$ $T(F_{pp_i}(t/2), F_{p_iq}(t/2)) \geq T(F_{pp_i}(\varepsilon/4), F_{p_iq}(\varepsilon/4)) \geq T(1-\eta, 1-\eta) > 1-\lambda'$ which implies that $D_{A_i}(\varepsilon) > 1 - \lambda' > 1 - \lambda$.

Conversely, by the Theorem 2.1 there exists $A_{\varepsilon\lambda} = \{p_1, \cdots, p_n\}$ such that for every $p \in S$ there exists $p_i = p_i(p) \in A_{\varepsilon\lambda}$ with $F_{pp_i}(\varepsilon) > 1 - \lambda$. Obviously $S = \bigcup_{i=1}^{n} S_{p_i}(\varepsilon, \lambda)$.

It is known that every metric space (S, d) is a probabilistic metric space with $F_{pq}(x) = H(x - d(p, q))$ where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

A characterization of the probabilistic precompact subsets is given by

THEOREM 2.4. *Let* $A \subseteq (S, F)$, *then* A *is a probabilistic precompact subset if and only if* $\alpha_A(x) = H(x)$.

Proof. Let $0 < \eta < 1$, $\varepsilon = x > 0$ and $\lambda = 1 - \eta$. Then there exists $\{A_i\}_{i \in I}$, with

$$D_{A_i}(\varepsilon) > 1 - \lambda = 1 - (1 - \eta) = \eta$$

and therefore $\eta \in K_{A,x}$, $\eta \leq \alpha_A(x)$. Since $\sup \eta \leq \alpha_A(x)$ it follows that $1 \leq \alpha_A(x)$ for every $x > 0$ and by the fact that $\alpha_A(x) \in \Delta$ we conclude that $\alpha_A(x) = H(x)$.

Conversely, if $\alpha_A(x) = H(x)$ we consider $\lambda > 0$ and $\varepsilon > 0$ then $\alpha_A(\varepsilon) = 1$ and there exists $\varepsilon_\lambda \in K_{A,\varepsilon}$ with the property $\alpha_A(\varepsilon) - \lambda < \varepsilon_\lambda$ and $1 - \lambda < \varepsilon_\lambda$. But $\varepsilon_\lambda \in K_{A,\varepsilon}$ which implies that there exists a family $\{A_i\}_{i\in I}$ with $D_{A_i}(\varepsilon) \geq \varepsilon_\lambda > 1 - \lambda$.

PROPOSITION 2.3. *Let* (S, d) *be a metric space and* (S, F_g) *the generated probabilistic metric space. Then*

$$\alpha_A(x) = H(x - \alpha(A)).$$

Proof. Let $B \subseteq S$; then for every $x > \delta(B) = \sup_{p,q \in B} d(p, q)$ we have $x > \inf_{p,q \in B} \{y : F_{pq}(y) = 1\}$ and therefore $F_{pq}(x) = 1$ for all $p, q \in B$ and $D_B(x) = \sup_{t<x} \{\inf_{p,q \in B} F(t)\} = 1$. If $x > \alpha(A)$ it follows that there exists $\{A_i\}_{i \in I}$ with $\delta(A_i) < x$ for all $i \in I$ and therefore $D_{A_i}(x) = 1 \geq \varepsilon$ for all ε, $0 < \varepsilon \leq 1$ and thus $\varepsilon \in K_{A,x}$ and $\alpha_A(x) = 1$. If $x = \delta(B)$ it is easy to see that $D_B(x) = 0$ and if $x < \delta(B)$ there exists $p, q \in B$ with the property that $x < d(p, q) = \inf\{y : F_{pq}(y) = 1\}$ and therefore $F_{pq}(x) < 1$ which implies that $F_{pq}(x) = 0$ and $D_B(x) = 0$. If $x < \alpha(A)$ it follows that there exists $\{A_i\}_{i \in I}$ with $\delta(A_i) \geq x$, $D_{A_i}(x) = 0$ and therefore for all x, $x \leq \alpha(A)$ and $\varepsilon > 0$ do not exist $\{A_i\}_{i \in I}$ with $D_{A_i}(x) \geq \varepsilon > 0$ and thus $\alpha_A(x) = 0$ for every $x \leq \alpha(A)$ and $\alpha_A(x) = H(x - \alpha(A))$.

COROLLARY. A *is a precompact set of* (S, d) *if and only if* A *is a probabilistic precompact set of* (S, F_g).

PROOF. It is known that A is a precompact set of (S, d) if and only if $\alpha(A) = 0$ and therefore $\alpha_A(x) = H(x)$ and by Theorem 2.4, A is a probabilistic precompact set. If A is a probabilistic precompact set in (S, F_g) we have $\alpha_A(x) = H(x)$ and by Proposition 2.3, $\alpha_A(x) = H(x - \alpha(A))$ which implies $\alpha(A) = 0$.

From this corollary and Theorem 2.3 it follows the importance of probabilistic precompacity introduced by Definition 2.3 and of Kuratowski function.

REFERENCES

[1] G. DARBO, *Punti uniti in trasformazioni a codominio non compatto*, « Rend. Sem. Mat. Padova », *24*, 84–92 (1955).

[2] R. J. EGBERT, *Products and quotients of probabilistic metric spaces*, « Pacific J. Math. », *24*, 437–455 (1960).

[3] M. FURI and A. VIGNOLI, *A fixed point theorem in complete metric spaces*, « Boll. U.M.I., SIV », *2*, 565–509 (1969).

[4] V. ISTRĂŢESCU and A. ISTRĂŢESCU, *On the theory of fixed points for some classes of mappings. I*, « Bull. Math. Soc. Sci. Math. Roum. », *14* (62), 419–426 (1970).

[5] V. ISTRĂŢESCU and A. ISTRĂŢESCU, *On the theory of fixed points for some classes of mappings. V*, « Atti Accad. Naz. Lincei », *51* (2), 162–167 (1971).

[6] C. KURATOWSKI, *Topologie*, 318 (1952).

[7] K. MENGER, *Statistical metrics*, « Proc. Nat. Acad. Sci. USA », *28*, 535–537 (1942).

[8] O. ONICESCU, *Nombres et systèmes aléatoires*, Éditions de l'Academie de la R. P. Roumaine, Bucarest.

[9] B. N. SADOVSKI, *A fixed point principle*, « Funct. Analiz. i priloj. », *1*, 74–76 (1967).

[10] B. SCHWEIZER and A. SKLAR, *Statistical metric spaces*, « Pacific J. Math. », *10*, 313–334 (1960).

[11] B. SCHWEIZER, A. SKLAR and E. THORP, *The metrization of statistical metric spaces*, « Pacific J. Math. », *10*, 673–675 (1960).

[12] J. R. L. WEBB, *A fixed point theorem and applications to functional equations in Banach spaces*, « Boll. U.M.I. », (4), 775–788 (1971).