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Recurrent Finsler space with Cartan's first curvature tensor field

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Geometria differenziale. — *Recurrent Finsler space with Cartan's first curvature tensor field.* Nota [(*)] di H. D. PANDE e T. A. KHAN, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — In questa Nota vengono definiti e studiati gli spazi ricorrenti di Finsler in rapporto al campo di tensori di curvatura secondo Cartan.

1. INTRODUCTION

Let F_n be an n-dimensional Finsler space with the positively homogeneous function $F(x, \dot{x})$ of degree one in $\dot{x}^i$. The metric tensor $g_{ij}(x, \dot{x})$ is given by $\frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$. The lowering and raising of indices are according to the notation, e.g. $g_{ir} M^i_{jk} = M_{jrk}$ and $g^{jr} M_{ijk} = M^r_{ik}$. The metric tensors are symmetric positively homogeneous function of degree zero in the directional argument. If $T^i_j(x, \dot{x})$ is any tensor, the Cartan's covariant derivative [2] [(1)] of second kind is given by

$$T^i_j|_h = F(\dot{\partial}_h T^i_j) + T^k_j A^i_{kh} - T^i_k A^k_{jh}, \tag{1.1}$$

where $A^i_{hj} = F(x, \dot{x})\, C^i_{hj}(x, \dot{x})$, is symmetric in its lower indices. We put $\dot{\partial}_i = \frac{\partial}{\partial \dot{x}^i}$ and $C_{ihj} = \frac{1}{2}\dot{\partial}_j g_{ih}(x, \dot{x})$ which is symmetric in all its lower indices. In n-dimensional Finsler space F_n, Cartan's curvature tensor of first kind is given by

$$S^i_{jkh} = A^i_{kr} A^r_{jh} - A^i_{rh} A^r_{jk}. \tag{1.2}$$

The commutative formulae for the tensor $S^i_{jkh}(x, \dot{x})$ are given by

$$T_{ij}|_{hk} - T_{ij}|_{kh} = F_{\dot{x}^k} T_{ij}|_h - F_{\dot{x}^h} T_{ij}|_k - T_{rj} S^r_{ikh} - T_{ir} S^r_{jkh}. \tag{1.3}$$

$$T^i_j|_{hk} - T^i_j|_{kh} = F_{\dot{x}^k} T^i_j|_h - F_{\dot{x}^h} T^i_j|_k + T^r_j S^i_{rkh} - T^i_r S^r_{jkh}. \tag{1.4}$$

and

$$F\dot{\partial}_l (S^i_{jkh}) - S^i_{jkh}|_l = S^i_{pkh} A^p_{jl} + S^i_{jph} A^p_{kl} + S^i_{jkp} A^p_{hl} - S^p_{jkh} A^i_{pl}. \tag{1.5}$$

The curvature tensor $S^i_{jkh}(x, \dot{x})$ satisfies the following identities

$$S^i_{jkh} + S^i_{khj} + S^i_{hjk} = 0. \tag{1.6}$$

$$S^i_{jhk} = - S^i_{jkh}. \tag{1.7}$$

$$S_{ijhk} = - S_{jihk}. \tag{1.8}$$

(*) Pervenuta all'Accademia il 26 settembre 1973.

(1) The numbers in the brackets refer to the references at the end of the paper.

(2) $F_{\dot{x}^m} = \dot{\partial}_m F = \partial F / \partial \dot{x}^m$.

The Cartan's first curvature tensor field S^i_{jkh} is said to be recurrent Cartan's curvature tensor field of the first order, if it satisfies the relation

$$S^i_{jkh}|_l = V_l \, S^i_{jkh}, \tag{1.9}$$

where V_l is a non-zero recurrent vector field [3].

2. Recurrent Cartan's first curvature tensor field of second order

In a n-dimensional Finsler space F_n if the curvature tensor $S^i_{jkh}(x, \dot{x})$ satisfies the relations

$$S^i_{jkh}(x, \dot{x})|_{lm} = a_{lm}(x, \dot{x}) \, S^i_{jkh}(x, \dot{x}) \tag{2.1}$$

and $(i, j, k \cdots = 1, 2, 3 \cdots, n)$

$$S^i_{jkh} \neq 0, \tag{2.2}$$

where a_{lm} is a tensor field dependent on the positional coordinates as well as directional, then it is said to be recurrent Finsler space of second order and a_{lm} is called a recurrent tensor field. The Cartan's first curvature tensor field of this space is defined as the recurrent Cartan's first curvature tensor field. In this paper we denote such a Finsler space by Fn^*.

THEOREM (2.1). *In a recurrent Finsler space of second order, the recurrent tensor field a_{lm} is non-symmetric.*

Proof. Commutating (2.1) with respect to the indices l and m we have,

$$S^i_{jkh}|_{lm} - S^i_{jkh}|_{ml} = (a_{lm} - a_{ml}) \, S^i_{jkh}. \tag{2.3}$$

Applying the commutation formula (1.4), we have,

$$F_{\dot{x}m} \, S^i_{jkh}|_l - F_{\dot{x}l} \, S^i_{jkh}|_m + S^r_{jkh} \, S^i_{rml} - S^i_{rkh} \, S^r_{jml} - S^i_{jrh} \, S^r_{kml} - \\ - S^i_{jkr} \, S^r_{hml} = (a_{lm} - a_{ml}) \, S^i_{jkh}. \tag{2.4}$$

It shows that a_{lm} is non-symmetric [1].

THEOREM (2.2). *In Fn^* the recurrent tensor field a_{lm} satisfies the following relation*

$$\{(a_{nm} - a_{mn}) \, F_{\dot{x}l} + (a_{ln} - a_{nl}) \, F_{\dot{x}m} + (a_{ml} - a_{lm}) \, F_{\dot{x}n}\} + \\ + \{V_l(a_{mn} - a_{nm}) + V_m(a_{nl} - a_{ln}) + V_n(a_{lm} - a_{ml})\} \\ = \{(a_{mn} - a_{nm})|_l + (a_{nl} - a_{ln})|_m + (a_{lm} - a_{ml})|_n\}. \tag{2.5}$$

Proof. Differentiating (2.4) covariantly with respect to x^n (in the sense of Cartan) and using relations (1.9), (2.1) and (2.4) we get,

$$\{(F_{\dot{x}m}|_n V_l - F_{\dot{x}l}|_n V_m + F_{\dot{x}m} \, a_{ln} - F_{\dot{x}l} \, a_{mn}) + V_n(a_{lm} - a_{ml}) + \\ + 2V_n(V_m \, F_{\dot{x}l} - F_{\dot{x}m} V_l)\} \, S^i_{jkh} = (a_{lm} - a_{ml})|_n \, S^i_{jkh}. \tag{2.6}$$

Making cyclic rotation with respect to the indices l, m, n in (2.6) and then adding all the three equations thus obtained, we get

$$
\begin{aligned}
(2.7) \qquad & [\{(a_{nm} - a_{mn}) F_{\dot{x}}^{\cdot l} + (a_{ln} - a_{nl}) F_{\dot{x}}^{\cdot m} + (a_{ml} - a_{lm}) F_{\dot{x}}^{\cdot n}\} + \\
& + \{V_l (a_{mn} - a_{nm}) + V_m (a_{nl} - a_{ln}) + V_n (a_{lm} - a_{ml})\} - \\
& - \{(a_{mn} - a_{nm})|_l + (a_{nl} - a_{ln})|_m + (a_{lm} - a_{ml})|_n\}] S^i_{jkh} = 0.
\end{aligned}
$$

By virtue of the recurrent properties of S^i_{jkh}, we obtain the result (2.5).

THEOREM (2.3). *In* Fn^*, *the Cartan's first curvature tensor* $S^i_{jkh}(x, \dot{x})$ *satisfies the relation*

$$
\begin{aligned}
(2.8) \quad & F_{\dot{x}}^{\cdot s} (a_{nl}|_m - a_{ml}|_n) + F_{\dot{x}}^{\cdot m} (a_{sl}|_n - a_{nl}|_s) + F_{\dot{x}}^{\cdot n} (a_{ml}|_s - a_{sl}|_m) + b_{mr} S^r_{lsn} + \\
& + b_{nr} S^r_{lms} + b_{sr} S^r_{lnm} = U_l \, \Phi_{\dot{x}}^{\cdot m}|_{ns} + F_{\dot{x}}^{\cdot l}|_s (\lambda_m|_n - b_{mn}) + \\
& + F_{\dot{x}}^{\cdot l}|_n (\lambda_s|_m - b_{sm}) + F_{\dot{x}}^{\cdot l}|_m (\lambda_n|_s - b_{ns}) + F_{\dot{x}}^{\cdot l} (b_{ms}|_n + b_{nm}|_s + b_{sn}|_m) + \\
& + (b_{lm} \lambda_n|_s + b_{ln} \lambda_s|_m + b_{ls} \lambda_m|_n) + V_n (b_{lm}|_s - b_{ls}|_m) + V_m (b_{ls}|_n - b_{ln}|_s) + \\
& + V_s (b_{ln}|_m - b_{lm}|_n) + 2 \{(\lambda_s|_n F_{\dot{x}}^{\cdot m} + \lambda_m|_s F_{\dot{x}}^{\cdot n} + \lambda_n|_m F_{\dot{x}}^{\cdot s}) V_l + \\
& + (V_l|_s F_{\dot{x}}^{\cdot n} - V_l|_n F_{\dot{x}}^{\cdot s}) V_m + (V_l|_m F_{\dot{x}}^{\cdot s} - V_l|_s F_{\dot{x}}^{\cdot m}) V_n + \\
& + (V_l|_s F_{\dot{x}}^{\cdot m} - V_l|_m F_{\dot{x}}^{\cdot n}) V_s\},
\end{aligned}
$$

where

$$
\begin{aligned}
b_{nm} &= a_{nm} - a_{mn}, \\
\lambda_m|_n &= V_m|_n - V_n|_m,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.9) \qquad U_l \, \Phi_{\dot{x}}^{\cdot m}|_{ns} = & \{V_l (F_{\dot{x}}^{\cdot m}|_{ns} - F_{\dot{x}}^{\cdot m}|_{sn}) - V_m (F_{\dot{x}}^{\cdot l}|_{ns} - F_{\dot{x}}^{\cdot l}|_{sn})\} + \\
& + \{V_l (F_{\dot{x}}^{\cdot n}|_{sm} - F_{\dot{x}}^{\cdot n}|_{ms}) - V_n (F_{\dot{x}}^{\cdot l}|_{sm} - F_{\dot{x}}^{\cdot l}|_{ms})\} + \\
& + \{V_l (F_{\dot{x}}^{\cdot s}|_{mn} - F_{\dot{x}}^{\cdot s}|_{nm}) - V_s (F_{\dot{x}}^{\cdot l}|_{mn} - F_{\dot{x}}^{\cdot l}|_{nm})\} .
\end{aligned}
$$

Proof. Differentiating (2.6) covariantly with respect to x^s and using equations (1.9) and (2.6), we obtain

$$
\begin{aligned}
(2.10) \quad & (F_{\dot{x}}^{\cdot m}|_{ns} V_l + F_{\dot{x}}^{\cdot m}|_n V_l|_s - F_{\dot{x}}^{\cdot l}|_{ns} V_m - F_{\dot{x}}^{\cdot l}|_n V_m|_s + F_{\dot{x}}^{\cdot m}|_s a_{ln} + F_{\dot{x}}^{\cdot m} a_{ln}|_s - \\
& - F_{\dot{x}}^{\cdot l}|_s a_{mn} - F_{\dot{x}}^{\cdot l} a_{mn}|_s) + V_n|_s (a_{lm} - a_{ml}) + V_n (a_{lm} - a_{ml})|_s + \\
& + 2V_n|_s (V_m F_{\dot{x}}^{\cdot l} - F_{\dot{x}}^{\cdot m} V_l) + 2V_n (V_m|_s F_{\dot{x}}^{\cdot l} + F_{\dot{x}}^{\cdot l}|_s V_m - F_{\dot{x}}^{\cdot m}|_s V_l - \\
& - F_{\dot{x}}^{\cdot m} V_l|_s) = (a_{lm} - a_{ml})|_{ns} .
\end{aligned}
$$

Interchanging the indices n and s in (2.10) and then subtracting from it and using the commutative formula (1.3), we get,

$$
\begin{aligned}
(2.11)\qquad & F_{\underset{x}{\cdot}s}(a_{lm}-a_{ml})|_n - F_{\underset{x}{\cdot}n}(a_{lm}-a_{ml})|_s - (a_{rm}-a_{mr})\, S^r_{lsn} - \\
& -(a_{lr}-a_{rl})\, S^r_{msn} = V_l(F_{\underset{x}{\cdot}m}|_{ns} - F_{\underset{x}{\cdot}m}|_{sn}) + V_m(F_{\underset{x}{\cdot}l}|_{sn} - F_{\underset{x}{\cdot}l}|_{ns}) + \\
& + (F_{\underset{x}{\cdot}m}|_n V_l|_s - F_{\underset{x}{\cdot}m}|_s V_l|_n) + (F_{\underset{x}{\cdot}l}|_s V_m|_n - F_{\underset{x}{\cdot}l}|_n V_m|_s) + \\
& + (F_{\underset{x}{\cdot}m}|_s\, a_{ln} - F_{\underset{x}{\cdot}m}|_n\, a_{ls}) + (F_{\underset{x}{\cdot}l}|_n\, a_{ms} - F_{\underset{x}{\cdot}l}|_s\, a_{mn}) + \\
& + F_{\underset{x}{\cdot}m}(a_{ln}|_s - a_{ls}|_n) + F_{\underset{x}{\cdot}l}(a_{ms}|_n - a_{mn}|_s) + (a_{lm}-a_{ml})(V_n|_s - V_s|_n) + \\
& + V_n(a_{lm}-a_{ml})|_s - V_s(a_{lm}-a_{ml})|_n + 2(V_m F_{\underset{x}{\cdot}l} - F_{\underset{x}{\cdot}m} V_l)\cdot \\
& \cdot(V_n|_s - V_s|_n) + 2\{(V_n V_m|_s - V_s V_m|_n)\, F_{\underset{x}{\cdot}l} + \\
& + (V_n F_{\underset{x}{\cdot}l}|_s - V_s F_{\underset{x}{\cdot}l}|_n)\, V_m + (F_{\underset{x}{\cdot}m}|_n V_s - F_{\underset{x}{\cdot}m}|_s V_n)\, V_l + \\
& + (V_s V_l|_n - V_n V_l|_s)\, F_{\underset{x}{\cdot}m}\}\,.
\end{aligned}
$$

Adding the equations obtained by cyclic interchange of the indices m, n and s in (2.11), we obtain the required result.

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