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Generalization of Meusnier's theorem in a Finsler space

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Geometria differenziale. — *Generalization of Meusnier's theorem in a Finsler space.* Nota (*) di CHANDRA MANI PRASAD, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Generalizzazione del teorema di Meusnier agli spazi di Finsler.

1. INTRODUCTION

Nirmala [2] obtained an extension of Meusnier's theorem by considering Singal-Ram Behari's [7] and Hsiao's [1] normal curvature of a congruence. In a Finsler space F_n, Prakash [4] has defined the generalised normal curvature of a curve relative to a congruence. It can be expressed in terms of the first curvature of the curve in F_n and certain angles, and the formula expressing this relation provides an extension of Meusnier's theorem. We derive a similar formula by considering the normal curvature of a congruence relative to another congruence.

Let the hypersurface F_{n-1} with local coordinates u^α, $\alpha = 1, \cdots, n-1$, be immersed in a Finsler space F_n with local coordinates x^i, $i = 1, 2, \cdots, n$. Consider a curve C: $x^i = x^i(s)$ on F_{n-1}, the components $x'^i = dx^i/ds$ and $u'^\alpha = du^\alpha/ds$ of its unit tangents at a point of F_{n-1} are related by

$$x'^i = B^i_\alpha u'^\alpha \tag{1.1}$$

where $B^i_\alpha = \partial x^i/\partial u^\alpha$ and the rank of the matrix $((B^i_\alpha))$ is $n-1$. The components g_{ij} and $g_{\alpha\beta}$ of the metric tensors of F_n and F_{n-1} are connected by the relation

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x') B^i_\alpha B^j_\beta \tag{1.2}$$

There exists a unit vector $n^i(u)$ called normal vector satisfying the conditions [6]

$$g_{ij}(x, n) n^j B^i_\alpha = n_i B^i_\alpha = 0 \tag{1.3}$$

$$g_{ij}(x, n) n^i n^j = 1 \tag{1.4}$$

The covariant derivative of B^i_α with respect to u^α is given by [6]

$$B^i_{\alpha;\beta} \overset{\text{def}}{=} I^i_{\alpha\beta} = \Omega_{\alpha\beta} n^i + \omega^i_{\alpha\beta} \tag{1.5}$$

where

$$\omega^i_{\alpha\beta} n_i = 0 \quad , \quad I^i_{\alpha\beta} n_i = \Omega_{\alpha\beta}$$

(*) Pervenuta all'Accademia il 26 settembre 1973.

and $\Omega_{\alpha\beta}(u, u')$ are the components of the second fundamental tensor of F_{n-1}. The covariant derivative of unit normal $n^i(u)$ is given by

$$n^i_{;\beta} = A^\delta_\beta B^i_\delta + V_\beta n^i \tag{1.6}$$

where

$$A^\delta_\beta = -\gamma^{\alpha\delta}\,\Omega_{\alpha\beta} - \gamma^{\alpha\delta} E_{ijk} B^k_\beta B^i_\alpha n^j$$

$$n^i_{;\beta} n_i = V_\beta$$

and we have written $E_{ijk} = g_{ij;k}(x, n)$.

2. Generalization of Meusnier's theorem

Consider the contravariant components λ^i tangent to a curve of the congruence which is such that through each point of the hypersurface one curve of each congruence passes. At a point of F_{n-1}, it is expressed as

$$\lambda^i = t^\alpha B^i_\alpha + cn^i \tag{2.1}$$

and we write

$$\lambda_i = g_{ij}(x, \lambda)\lambda^j, \quad \lambda_i \lambda^i = 1. \tag{2.2}$$

Considering the δ-derivative [6] of (2.1) and first of (2.2) and simplifying, we have

$$\frac{\delta\lambda_i}{\delta s} = \lambda_{i};\beta\,\frac{du^\beta}{ds} = (q^\alpha_{ij\beta} B^j_\alpha + p_{ij\beta} n^j + g_{ij}(x, \lambda)\, t^\alpha \Gamma^j_{\alpha\beta})\,\frac{du^\beta}{ds} \tag{2.3}$$

where

$$q^\alpha_{ij\beta} = \bar{E}_{ij\beta} t^\alpha + g_{ij}(x, \lambda)\,[t^\alpha_{;\beta} + cA^\alpha_\beta] \tag{2.4}$$

$$p_{ij\beta} = c\bar{E}_{ij\beta} + g_{ij}(x, \lambda)\,[c;\beta + cV_\beta] \tag{2.5}$$

and we have taken $\bar{E}_{ij\beta} = g_{ij;\beta}(x, \lambda)$.

In analogy to the definition of normal curvature [6], Prakash [4] has defined the generalised normal curvature of a curve relative to the congruence λ^i as

$$-x'^i(\delta\lambda_i/\delta s)$$

and has denoted by $[R_\lambda(x, x')]^{-1}$. Thus

$$R^{-1}_\lambda(x, x') = -x'^i(\delta\lambda_i/\delta s) = \hat{\Omega}_{\beta\gamma} u'^\beta u'^\gamma \tag{2.6}$$

where

$$\hat{\Omega}_{\beta\gamma} = -\{q^\alpha_{ij\beta} B^j_\alpha + p_{ij\beta} n^j + g_{ij}(x, \lambda)\, t^\alpha \Gamma^j_{\alpha\beta}\} B^i_\gamma. \tag{2.7}$$

Let the angle between the unit vectors λ^i and x'^i be φ_0, then

$$\cos\varphi_0 = g_{ij}(x, \lambda)\lambda^j x'^i = \lambda_i x'^i. \tag{2.8}$$

The δ-derivative of (2.8) with respect to C gives

(2.9) $$x'^i \frac{\delta\lambda_i}{\delta s} = -\frac{\cos\varphi_1}{r} - \sin\varphi_0 \frac{d\varphi_0}{ds}$$

where φ_1 is the angle between the vectors λ^i and $\delta x'^i/\delta s$, and

$$\frac{1}{\gamma} = \left[g_{ij}(x, \delta x'/\delta s) \frac{\delta x'^i}{\delta s} \frac{\delta x'^j}{ds} \right]^{1/2}$$

is the first curvature of the curve C in F_n (Rund [6], pp. 152).

By virtue of the equation (2.6) and (2.9), we write

(2.10) $$\frac{1}{R_\lambda(x, x')} = \frac{\cos\varphi_1}{r} + \sin\varphi_0 \frac{d\varphi_0}{ds}.$$

PARTICULAR CASES. (i) If the curve C is a union curve [8, 9],

$$\varphi_1 = \frac{\pi}{2} - \varphi_0$$

$$\cos\varphi_1 = \sin\varphi_0 .$$

Hence the equation (2.10) reduces to

$$\frac{1}{R_\lambda} = \left(\frac{1}{r} - \frac{d\varphi_1}{ds}\right) \cos\varphi_1$$

(ii) If the congruence λ^i is normal to F_{n-1},

$$t^\alpha = 0 \quad , \quad c = 1,$$

so that

$$\cos\varphi_0 = x'^i n_i = 0$$

and the generalized normal curvature of a curve relative to a congruence λ^i reduces to the normal curvature of F_{n-1} (Rund [6], page 192). The equation (2.10) then takes the form

(2.11) $$1/R = (1/r) \cos\varphi_1$$

which is Meusnier's theorem (Rund [6], pp. 192). Hence (2.10) *is a generalization of* Meusnier's *theorem.*

Consider another congruence μ^i which, at a point of F_{n-1}, is expressed as

(2.12) $$\mu^i = p^\alpha B^i_\alpha + \Gamma n^i$$

We define the normal curvature of a congruence λ^i with respect to another congruence μ^i as

$$\mu^i (\delta\lambda_i/\delta s)/[g_{\alpha\beta}(u, t)\, t^\alpha t^\beta]^{1/2}$$

and denote it by ${}_\mu R_\lambda^{-1}(x, x')$. Now using the equations (2.3) and (2.12) and after simplification, we have

(2.13) $$t \cdot {}_\mu R_\lambda^{-1} = \mu^i \frac{\delta\lambda_i}{\delta s} = \hat{\Omega}_\beta \frac{du^\beta}{ds} - \hat{\Omega}_{\beta\gamma} \frac{du^\beta}{ds} p^\gamma$$

where

$$t = [g_{\alpha\beta}(u, t)\, t^{\alpha} t^{\beta}]^{1/2}$$

and

$$\hat{\Omega}_{\beta} = \Gamma\, [q_{ij\beta}\, B^{j}_{\alpha} n^{i} + p_{ij\beta}\, n^{i} n^{j} + g_{ij}(x, \lambda)\, t^{\alpha} n^{i}\, I^{j}_{\alpha\beta}] . \tag{2.14}$$

If the spaces F_n and F_{n-1} are Riemannian, the equation (2.13) gives the normal curvature of the congruence λ^i relative to another congruence for a hypersurface of a Riemannian space (Hsiao [1]).

Let the angle between the unit vectors λ^i and μ^i be denoted by θ_0, then

$$\cos\theta_0 = \lambda_i\, \mu^i. \tag{2.15}$$

If we denote the angle between λ^i and $\delta\mu^i/\delta s$ by θ_1, and write

$$r_{\mu}^{-1} = \left[g_{ij}\left(x, \frac{\delta\mu}{\delta s}\right) \frac{\delta\mu^i}{\delta s} \frac{\delta\mu^j}{\delta s} \right]^{1/2} \tag{2.16}$$

then

$$r_{\mu}^{-1} \cos\theta_1 = \lambda_i \frac{\delta\mu^i}{\delta s} . \tag{2.17}$$

Taking the δ-derivative of (2.15) and using (2.13) and (2.17) we get

$$\frac{t}{{}_{\mu}R_{\lambda}} = -\left[\frac{\cos\theta_1}{r_{\mu}} + \sin\theta_0 \frac{d\theta_0}{ds} \right] . \tag{2.18}$$

The equation (2.18) *relates the normal curvature of a congruence* λ^i *with respect to another congruence* μ^i *and absolute curvature* (*first curvature*) *of the congruence* μ^i *in* F_n.

PARTICULAR CASES. (i) If the curve C is an absolute geodesic [3, 5] of the congruence μ^i,

$$1/r_{\mu} = 0 ,$$

and then

$$\frac{t}{{}_{\mu}R_{\lambda}} = \frac{d}{ds}(\cos\theta_0)$$

Hence *the normal curvature of a congruence* λ^i *relative to the congruence* μ^i *is proportional to the derivative of the cosine of the angle between the vectors* λ^i *and* μ^i.

(ii) When μ^i is tangential to the curve C, we have

$$\Gamma = 0 \quad , \quad p^{\alpha} = du^{\alpha}/ds ,$$

and then $\hat{\Omega}_{\beta} = 0$ and $1/r_{\mu} = 1/r$. From (2.13) it follows that

$$t\, [{}_{\mu}R_{\lambda}]^{-1} = -\, \hat{\Omega}_{\alpha\beta}\, u'^{\alpha} u'^{\beta} = -\frac{1}{R_{\lambda}} .$$

Hence the equation (2.18) reduces to

$$\frac{1}{R_\lambda} = \left[\frac{\cos \varphi_1}{r} + \sin \varphi_0 \frac{d\varphi_0}{ds}\right]$$

which is identical with the equation (2.10), and so (2.10) is a particular case of (2.18). *Thus* (2.18) *is another generalization of* Meusnier's *theorem.*

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